Presentation and Structure of Substitutes Valuations

[Extended Abstract]*

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ABSTRACT

We propose two different methods for presenting substitutes (a.k.a. gross-substitutes) valuations. Each provides short descriptions for a family of substitutes valuations. We also show that substitutes valuation are closed under k-satiation.

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1. BACKGROUND

Given a finite set of items Ω, a valuation is a function v that associates a real number to every subset of Ω. We shall require that valuations are normalized: $v(\emptyset) = 0$ and monotonic: $v(A) \geq v(B)$ if $A \supseteq B$. In [3], Kelso and Crawford introduced the notion of a *substitutes* valuation for profit-maximizing firms in a model with discrete inputs (workers). The substitutes property mandates that increasing the price of an item will not decrease the demand for any other item: i.e., if an item i is in a preferred set at prices $\vec{p}$, then increasing p_j for some $j \neq i$, will still have item i in some preferred set. Theirs and subsequent [2, 5] works imply that several problems of economic importance can be solved in an elegant and effective way when the agents exhibit valuations that are substitutes. In all of the mechanisms proposed, agents must describe their substitutes valuation. It is therefore important to find intuitive and concise formalisms that will enable agents to describe such valuations, guaranteeing that such valuations are indeed substitutes. In [1], Fujishige and Yang proposed a very powerful characterization of substitutes valuations as M#-concave functions. Most notable are the facts that substitutes valuations are closed under convolution, also called OR, and that, given prices for the items, the problem of finding a preferred bundle under those prices can be solved in polynomial time in the oracle model, that the problem of finding an optimal allocation amongst substitutes agents is solvable in polynomial time and that substitutes valuations have zero measure amongst submodular valuations. Notice, though, that no polynomial time algorithm is known that can check in the oracle model whether a valuation is substitutes.

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2. S-PRESENTATIONS

We present a valuation as a sum of symmetric valuations and therefore such presentations are called S-presentations. S-presentations define S-valuations which form a strict subfamily of GS, the family of substitutes valuations. We recall that a symmetric valuation is substitutes iff it is submodular, i.e., concave.

Intuitively, consider an employment service. The potential employees come with a natural tree-structure, i.e., a hierarchy. For example the potential employees can be classified into: doctors, nurses and cleaners. Each of those may be further classified into: male nurses and female nurses for example, or cardiologists, oncologists and pediatricians. The value of a given team of employees for the firm may be evaluated as a (submodular) function of the number of doctors, plus a (submodular) function of the number of nurses, plus a (submodular) function of the number of cleaners minus some (supermodular) function of the total size of the team accounting for the cost of processing them.

A family $\mathcal{H}$ of subsets of X is a hierarchy iff, for any $h, h' \in \mathcal{H}$ such that $h \cap h' \neq \emptyset$, either $h \subseteq h'$ or $h' \subseteq h$. We assume that a hierarchy on X is given and that, for every $h \in \mathcal{H}$, a *symmetric submodular valuation* on h, v_h is given. Note that v_h is substitutes. Note that $v_h(A)$ depends only on $\mid A \cap h \mid$, the cardinality of $A \cap h$. The function $A \mapsto v_h(A)$ is a GS valuation. The following lemma spells out the particular property of such valuations.

DEFINITION 1. *Any hierarchy $\mathcal{H}$ and symmetric submodular valuations v_h on h define a valuation v by:*

$$v(A) = \sum_{h \in \mathcal{H}} v_h(A). \tag{1}$$

A function v defined in such a way, for some hierarchy $\mathcal{H}$ and some symmetric submodular valuations v_h, is obviously a valuation and will be said to be an S-valuation.

THEOREM 1. *Any S-valuation is substitutes.*

We found the result above had been obtained by Murota [7] (see p. 141, laminar convex functions). In Theorem 1 the assumption that $\mathcal{H}$ is a hierarchy cannot be dispensed with. Not all GS valuations are S-valuations: a counter-example has been found but cannot be presented here for lack of space. Indeed the counter-example shows an H-valuation (see Section 4) that is not an S-valuation.

3. K-SATIATION

The k-satiation of a valuation v is the valuation v_k that associates to any bundle A the maximal value $v(B)$ amongst all subsets $B \subseteq A$ that contain no more than k elements: $v_k(A) = \max_{B \subseteq A, |B| \leq k} v(B)$. In [2], Gul and Stacchetti noticed that the k-satiations of additively separable or additively concave valuations were substitutes (GS). We prove the more general result: the class of substitutes valuations (GS) is closed under satiation.

Theorem 2. *If v is a substitutes valuation, its k-satiation, v_k is substitutes, for any k.*

4. H-PRESENTATIONS

Matching algorithms such as those described in [6] assume that firms can describe their valuations for bundles of potential employees (or contracts) and that those valuations are substitutes. We shall present here a way firms may describe complex valuations that are guaranteed to be substitutes: H-presentations.

Consider a simplified version of the recruitment problem faced by a hospital. The problem is to attach a numerical value to any subset of a pool of potential employees (all employees will be called doctors for simplification). The hospital comprises a number of basic units, such as Pediatrics, Internal Medecine1, Internal Medecine2, etc ... Those basic units form a hierarchy, such as: Hospital, Building1, Building2, West wing of Building1, East wing of Building2, etc ... Each of the potential employees has a specific utility in each of the *basic* units: Dr. White has value 10 in Pediatrics, but only value 2 in Obstetrics. Each of the units in the hierarchy, including the basic units and the whole hospital have a limit on the number of doctors they can employ: not more than 132 doctors in the West wing of Building3. An allocation of doctors to the *basic* units is feasible if those limits are all satisfied. The value of an allocation of doctors to the *basic* units is the sum of the utilities of the individual doctors in the unit they are assigned to.

The value of any set of doctors is the value of the best feasible allocation of those doctors. Such a presentation of a valuation will be called an H-presentation. Notice that any description of the type described above is short, i.e. polynomial in the number of doctors and basic units. The recruitment problem, and the valuation function, may be described by an Integer Programming problem. Let D be the set of doctors, U the set of basic units and $\mathcal{H} \subseteq 2^U$ is a hierarchy. Given a^i_j the value of doctor i in unit j, and a maximal size b_h for every $h \in \mathcal{H}$,

Maximize $\sum_{i \in D, j \in U} x^i_j \, a^i_j$, under the constraints:

1. $\sum_{j \in h} \sum_{i \in D} x^i_j \leq b_h$ for every $h \in \mathcal{H}$,
2. $\sum_{j \in U} x^i_j \leq 1$ for every $i \in D$, and
3. $x^i_j \in \{0, 1\}$.

The first constraint expresses that no more than b_h doctors may work in the basic units of wing h and the second constraint expresses that a doctor may work in at most one basic unit.

Theorem 3. *The valuation defined an H-presentation is substitutes.*

This follows from the fact that, neglecting the number limitations, any basic unit defines a linear (OS in the language of [4]) valuation, therefore substitutes. The number limitations on the basic units define k-satiations of substitutes (here linear) valuations, therefore substitutes valuations, by Theorem 2. The juxtaposition of a number of units defines a valuation that is the convolution (OR in the language of [4]) of the valuations for the specific units. By Theorem 5, there, GS is closed under OR and this valuation is substitutes. Again, applying number limitations boils down to considering the k-satiation of a substitutes valuation and leaves us with a substitutes valuation.

An H-presentation defines the value of a bundle as the solution of an optimization problem: find the value of the best feasible allocation. This optimization problem is equivalent to a weighted max-flow problem: a max-flow problem in which edges have both a capacity and a weight and one wants to maximize the weight of the flow, subject to the capacity constraints. Such problems have integral optimal solutions and can be solved in polynomial time.

We have been able to find some S-valuation that can be shown not to be an H-valuation, thus separating the two classes and showing that not all substitutes valuations are H-valuations.

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