Optimal Incentives in Core-Selecting Auctions
Robert Day, School of Business, University of Connecticut
Paul Milgrom, Dept of Economics, Stanford University
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Auctions that select core allocations with respect to reported values generate competitive levels of sales revenues at equilibrium and limit bidder incentives to use shills. Among core-selecting auctions, the ones that select bidder-optimal points in the core are efficient in minimizing the bidders’ incentives to misreport. Those auctions select the Vickrey outcome whenever that lies in the core and otherwise select points with higher than Vickrey revenues. Among core allocations, the points that minimize total seller revenue also minimize the sum of the gains that bidders can earn from unilateral deviations. Minimum-revenue core-selecting auctions have recently been planned or implemented for several high-stakes applications.

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I. Introduction
In early 2008, the United Kingdom’s telecommunication authority, Ofcom, adopted a new pricing rule for its spectrum auction – a minimum-revenue core-selecting rule. The class of such rules had only recently been proposed and analyzed by Day and Milgrom (2007). Following the UK’s lead, radio spectrum auctions with similar rules were planned in Austria, Denmark, Ireland,

1 This paper updates and corrects work that we originally reported in Day and Milgrom (2007). Our text borrows liberally from our own earlier work.
Portugal, and the Netherlands, and by the Federal Aviation Administration in the United States for the allocation of landing slot rights at New York City airports.\(^2\)

The new pricing rule generalizes the familiar second-price auction rule for auctions of a single item. One way to characterize the outcome of a second-price auction is in terms of the \textit{core}: the price is high enough that no bidder (or group of bidders) is willing to offer more to the seller to change the assignment and, among such prices, it is the lowest one. For multi-item auctions, a core price vector is one that is low enough to be individually rational and high enough that no group of bidders finds it profitable to offer a higher total price to the seller. Among core price vectors, the minimum-revenue core vectors are the ones with the smallest revenue for the seller.

Two general considerations inspired our development of the theory of core prices and core-selecting auctions. The first was discontent with the auction proposed by Vickrey (1961), whose weaknesses are reviewed by Ausubel and Milgrom (2006). Of particular concern is that Vickrey prices can be very low. The second was that similar core and stability concepts have been applied successfully in the design of real-world matching markets. The National Resident Matching Program is a famous example, but others include the mechanisms adopted by New York and Boston for assigning students to schools and the New England Kidney Exchange (Roth and Peranson 1999), Roth, Sonmez and Unver (2005), Abdulkadiroglu, Pathak, Roth and Sonmez (2005a, b).

There is both empirical and experimental evidence to suggest that the core is important, although most has focused on matching rather than on auctions. Stable matching mechanisms survive much longer in practical applications than related unstable mechanisms (Roth and Xing (1994), Kagel and Roth (2000)). And there is a theoretical argument to explain this longevity: if

\(^2\) Most of these auctions also incorporated multiple rounds of bids following a suggestion of Ausubel, Cramton and Milgrom (2006).
a proposed match is stable, then no group would prefer to renege and make an alternative arrangement among themselves, because there is no feasible alternative that all group members would prefer. But if a proposed match is unstable, then some group would prefer to renege, and too much reneging would make the mechanism unreliable for its users.

Nothing limits this theoretical argument to the case matching. For an auction, if a mechanism produces a core allocation, then no group of bidders can profitably offer a higher total price to the seller. And if the auction selects a point that is not in the core at least with respect to the submitted bids, then some group of bidder has already offered the seller a total price that is higher than the price prescribed by the auction. It is easy to see why sellers might want to renege and make a separate deal with that group of bidders.

Parts of these analyses assume that the recommended matching or auction mechanisms result in stable or core allocations, but whether that happens depends on the participants’ strategies. Participant behavior in real mechanisms varies widely from naïve to sophisticated, and the most sophisticated participants do not merely make truthful reports in the mechanism. Instead, they also make decisions about whether to make pre-emptive offers before the auction, to enter the auction as a single bidder or as several, to stay out of the auction and try to bargain with the winners afterwards, to buy extra units in the auction and resell some afterwards, to renege on deals, or to persuade the seller to make changes to the timing or rules of the mechanism. Each of these elements can be important in some auction settings.

Despite the variety of objectives and of important behavioral constraints in real auction settings, mechanism design researchers customarily impose truth-telling incentives first and then to ask to what extent other objectives or constraints can be accommodated. Since optimization is

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3 The core is always non-empty in auction problems. Indeed, for any profile of reports, the allocation that assigns the items efficiently and charges each bidder the full amount of its bids selects a core allocation. This selection describes the “menu auction” analyzed by Bernheim and Whinston (1986). Other core-selecting auctions are described in Ausubel and Milgrom (2002) and Day and Raghavan (2007).
at best an approximation to the correct behavioral theory for bidders, it is also interesting to
reverse the exercise, asking: by how much do the incentives for truthful reporting fail when other
design objectives are treated as constraints?

The modern literature does include some attempts to account for multiple performance
criteria even when incentives are less than perfect. Consider, for example, the basic two-sided
matching problem, commonly called the marriage problem, in which men have preferences over
women and women have preferences over men. The early literature treats stability of the outcome
as the primary objective, and only later turns its attention to the incentive properties of the
mechanism. In the marriage problem, there always exists a unique man-optimal match and a
unique woman-optimal match. The direct mechanism that always selects the man-optimal match,
is strategy-proof for men but not for women and the reverse is true for the woman-optimal
mechanism. Properties such as these are typically reported as advantages of the mechanism, even
though these incentives fall short of full strategy-proofness. Another argument is that even when
strategy-proofness fails, finding profitable deviations may be so hard that most participants find it
best just to report truthfully. A claim of this sort has been made for the pre-1998 algorithm used
by National Resident Matching Program, which was not strategy-proof for doctors, but for which
few doctors could have gained at all by misreporting and for which tactical misreporting was
fraught with risks (Roth and Peranson (1999)).

The analysis of multiple criteria is similarly important for the design of package auctions
(also called “combinatorial auctions”), which are auctions for multiple items in which bidders can

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4 As Gale and Shapley (1962) first showed, there is a stable match that is Pareto preferred by all men to any
other stable match, which they called the “man optimal” match.

5 Hatfield and Milgrom (2005) identify the conditions under which strategy-proofness extends to cover the
college admissions problem, in which one type of participant (“colleges”) can accept multiple applicants,
but the other kind (“students”) can each be paired to only one college. Their analysis also covers problems
in which wages and other contract terms are endogenous.

6 For example, see Abdulkadiroglu, Pathak, Roth and Sonmez (2005a).

7 There is quite a long tradition in economics of examining approximate incentives in markets, particularly
when the number of participants is large. An early formal analysis is by Roberts and Postlewaite (1976).
bid directly for non-trivial subsets (“packages”) of the items being sold, rather than being restricted to submit bids on each item individually. In these auctions, revenues are an obvious criterion. Auctions are commonly run by an expert auctioneer on behalf of the actual seller and any failure to select a core allocation with respect to reported values implies that there is a group of bidders who have offered to pay more in total than the winning bidders, yet whose offer has been rejected. Imagine trying to explain such an outcome to the actual seller or, in a government sponsored auction, to a skeptical public? Another possible design objective is that a bidder should not profit by entering and playing as multiple bidders, rather than as a single one.

We illustrate these conditions and how they fail in the Vickrey auction with an example of two identical items are for sale. The first bidder wants both items and will pay up to 10 for the pair; it has zero value for acquiring a single item. The second and third bidders each have values of 10 for either one or two items, so their marginal values of a second item are zero. The Vickrey auction outcome assigns the items to the second and third bidders for prices of zero. Given that any of the three bidders would pay 10 for the pair of items, a zero price is surely too low: that is the low revenue problem. Generally, the low revenue problem for the Vickrey auction is that its payments to the seller may be less than those at any core allocation. Also, suppose that the second and third bidders are both controlled by the same player whose actual values are 10 for one item or 20 for two. If the bidder were to participate as a single entity, it would win the two items and pay a price of ten. By bidding as two entities, each of which demands a single item for

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8 McMillan (1994) describes how heads rolled when second-price auctions were used to sell spectrum rights in New Zealand and the highest bid was sometimes orders of magnitude larger than the second highest bid.

9 Yokoo, Sakurai and Matsubara (2004) were the first to emphasize the importance of “false name bidding” and how it could arise in the anonymous environment of Internet auctions. The problem they identified, however, is broader than just anonymous Internet auctions. For example, in the US radio spectrum auctions, several of the largest corporate bidders (including AT&T, Cingular, T-Mobile, Sprint, and Leap Wireless) have at times had contracts with or financial interests in multiple bidding entities in the same auction, enabling strategies that would not be possible for a single, unified bidder.

10 In this example, the core outcomes are the outcomes in which 2 and 3 are the winning bidders, each pays a price between zero and ten, and the total payments are at least ten. The seller’s revenue in a core-selecting auction is thus at least 10.
a price of 10, the player reduces its total Vickrey price from ten to zero: that is the shill bidding
problem. These vulnerabilities are so severe that practical mechanism designers are compelled to
investigate when and whether relaxing the incentive compatibility objective can alleviate these
problems.

We have discussed matching and package auction mechanisms together not only because
they are two of the currently mostly active areas of practical mechanism design but also because
there are some remarkable parallels between their equilibrium theories. One parallel connects the
cases where the doctors in the match are substitutes for hospital and when the goods in the
auction are substitutes for the bidders. In these cases, the mechanism that selects the doctor-
optimal match is *ex post* incentive-compatible for doctors and a mechanism, the ascending proxy
auction of Ausubel and Milgrom (2002), which selects a bidder-optimal allocation (a core
allocation that is Pareto optimal for bidders), is *ex post* incentive-compatible for bidders.\(^{11}\)

A second important connection is the following one: for every stable match \(x\) and every
stable matching mechanism, there exists an equilibrium in which each player adopts a certain
*truncation strategy*, according to which it truthfully reports its ranking of all the outcomes at
which it is not matched, but reports that it would prefer to be unmatched rather than to be
assigned an outcome worse than \(x\). What is remarkable about this theorem is that *one single
profile of truncation strategies is a Nash equilibrium for every stable matching mechanism*. We
will find that a similar property is true for core-selecting auctions, but with one difference. In
matching mechanisms, it is usual to treat all the players are strategic, whereas in auctions it is not
uncommon to treat the seller differently, with only a subset of the players—the *bidders*—treating
as making decisions strategically. We are agnostic about whether to include the seller as a bidder
or even whether to include all the buyers as strategic players. Regardless of how the set of
strategic players is specified, we find that for every allocation on the Pareto-frontier of the core

\(^{11}\) This is also related to results on wage auctions in labor markets as studied by Kelso and Crawford (1982)
and Hatfield and Milgrom (2005), although those models do not employ package bidding.
for the players who report strategically, there is a single profile of truncation strategies that is an
equilibrium profile for every core-selecting auction.\textsuperscript{12}

The preceding results hinge on another similarity between package auctions and matching
mechanisms. In any stable matching mechanism or core-selecting auction and given any reports
by the other players, a player’s best reply achieves its maximum core payoff or best stable match
given its actual preferences and the reported preferences of others. For auctions, there is an
additional interesting connection: the maximum core payoff is exactly the Vickrey auction
payoff.

Next are the inter-related results about incentives for groups of participants. Given a core-
selecting auction, the incentives for misreporting are minimal for individuals in a particular group
$S$ if and only if the mechanism selects an $S$-best core allocation. If there is a unique $S$-best
allocation, then truthful reporting by members of coalition $S$ is an ex post equilibrium. This is
related to the famous result from matching theory (for which there always exists a unique man-
optimal match and a unique woman-optimal match) that it is an ex post equilibrium for men to
report truthfully in the man-optimal mechanism and for women to report truthfully in the woman-
optimal mechanism.

The remainder of this paper is organized as follows. Section II formulates the package
auction problem. Section III characterizes core-selecting mechanisms in terms of revenues that
are never less than Vickrey revenues, even when bidders can use shills. Section IV introduces
definitions and notation and introduces the theorems about best replies and full information
equilibrium. Section V states and proves theorems about the core-selecting auctions with the
smallest incentives to misreport. Various corresponding results for the marriage problem are
developed in section VI. Section VII notes an error regarding revenue monotonicity in an earlier

\textsuperscript{12} These truncation strategies also coincide with what Bernheim and Whinston (1986) call “truthful
strategies” in their analysis of a “menu auction,” which is a kind of package auction.
version of this chapter as it appeared in the Int. Journal of Game Theory, and makes connections
to more recent research and applications. Section VIII concludes.

II. Formulation

We denote the seller as player 0, the bidders as players \( j = 1, \ldots, J \), and the set of all players
by \( N \). Each bidder \( j \) has quasi-linear utility and a finite set of possible packages \( X_j \). Its value
associated with any feasible package \( x_j \in X_j \) is \( u_j(x_j) \geq 0 \). For convenience, we formulate our
discussion mainly in terms of bidding applications, but the same mathematics accommodates
much more, including some social choice problems. In the central case of package bidding for
predetermined items, \( x_j \) consists of a package of items that the bidder may buy. For procurement
auctions, \( x_j \) could also usefully incorporate information about delivery dates, warranties, and
various other product attributes or contract terms. Among the possible packages for each bidder is
the null package, \( \emptyset \in X_j \) and we normalize so that \( u_j(\emptyset) = 0 \).

For concreteness, we focus on the case where the auctioneer is a seller who has a feasible
set \( X_0 \subset X_1 \times \cdots \times X_J \) with \( (\emptyset, \ldots, \emptyset) \in X_0 \)—so the no sale package is feasible for the seller—and
a valuation function \( u_0 : X_0 \to \mathbb{R} \) normalized so that \( u_0(\emptyset, \ldots, \emptyset) = 0 \). For example, if the seller
must produce the goods to be sold, then \( u_0 \) may be the auctioneer-seller’s variable cost function.

For any coalition \( S \), a goods assignment \( \hat{x} \) is feasible for coalition \( S \), written \( \hat{x} \in F(S) \), if
(1) \( \hat{x} \in X_0 \) and (2) for all \( j \), if \( j \not\in S \) or \( 0 \not\in S \), then \( \hat{x}_j = \emptyset \). That is, a bidder can have a non-null
assignment when coalition \( S \) forms only if that bidder and the seller are both in the coalition.

The coalition value function or characteristic function is defined by:

\[
w_u(S) = \max_{x \in F(S)} \sum_{j \in S} u_j(x_j)
\]  

(1)
In a direct auction mechanism \((f, P)\), each bidder \(j\) reports a valuation function \(\hat{u}_j\), and the profile of reports is \(\hat{u} = \{\hat{u}_j\}_{j=1}^t\). The outcome of the mechanism, \(\left( f(\hat{u}), (P_j(\hat{u})) \right) \in (X_0, \mathbb{R}_+^t)\), specifies the choice of \(x = f(\hat{u}) \in X_0\) and the payments \(p_j = P_j(\hat{u}) \in \mathbb{R}_+\) made to the seller by each bidder \(j\). The associated payoffs are given by \(\pi_0 = u_0(x) + \sum_{j \neq 0} p_j\) for the seller and \(\pi_j = u_j(x) - p_j\) for each bidder \(j\). The payoff profile is individually rational if \(\pi \geq 0\).

A cooperative game (with transferable utility) is a pair \((N, w)\) consisting of a set of players and a characteristic function. A payoff profile \(\pi\) is feasible if \(\sum_{j \in N} \pi_j \leq w(N)\), and in that case it is associated with a feasible allocation. An imputation is a feasible, non-negative payoff profile. An imputation is in the core if it is efficient and unblocked:

\[
\text{Core}(N, w) = \left\{ \pi \geq 0 \mid \sum_{j \in N} \pi_j = w(N) \text{ and } \left( \forall S \subseteq N \right) \sum_{j \in S} \pi_j \geq w(S) \right\}
\]

A direct auction mechanism \((f, P)\) is core-selecting if for every report profile \(\hat{u}\), \(\pi_{\hat{u}} \in \text{Core}(N, w_{\hat{u}})\). Since the outcome of a core-selecting mechanism must be efficient with respect to the reported preferences, we have the following:

**Lemma 1.** For every core-selecting mechanism \((f, P)\) and every report profile \(\hat{u}\),

\[
f(\hat{u}) \in \arg \max_{x \in X_0} \sum_{j \in N} \hat{u}_j(x_j)
\]

The payoff of bidder \(j\) in a Vickrey auction is the bidder’s marginal contribution to the coalition of the whole. In cooperative game notation, if the bidders’ value profile is \(u\), then bidder \(j\)’s payoff is \(\pi_j = w_u(N) - w_u(N - j)\).\(^{13}\)

\(^{13}\) A detailed derivation can be found in Milgrom (2004).
III. Revenues and Shills: Necessity of Core-Selecting Auctions

We have argued that the revenues from the Vickrey outcome are often too low to be acceptable to auctioneers. In order to avoid biasing the discussion too much, in this section we treat the Vickrey revenues as a just-acceptable lower bound and ask: what class of auctions have the properties that, for any set of reported values, they select the total-value maximizing outcome and lead always to bidder payoffs no higher than the Vickrey payoffs, even when bidders may be using shills? Our answer will be: exactly the class of core-selecting auctions.

In standard fashion, we call any mechanism with the first property, namely, that the auction selects the total-value-maximizing outcome, “efficient.”

**Theorem 1.** An efficient direct auction mechanism has the property that no bidder can ever earn more than its Vickrey payoff by disaggregating and bidding with shills if and only if it is a core-selecting auction mechanism.

**Proof.** Fix a set of players (seller and bidders) \( N \), let \( w \) be the coalitional value function implied by their reported values, and let \( \pi \) be the players’ vector of reported payoffs. Efficiency means \( \sum_{j \in N} \pi_j = w(N) \). Let \( S \subseteq N \) be a coalition that excludes the seller. These bidders could be shills. Our condition requires that they earn no more than if they were to submit their merged valuation in a Vickrey auction, in which case the merged entity would acquire the same items and enjoy a total payoff equal to its marginal contribution to the coalition of the whole: \( w(N) - w(N - S) \). Our restriction is therefore \( \sum_{j \in S} \pi_j \leq w(N) - w(N - S) \). In view of efficiency, this holds if and only if \( \sum_{j \in N - S} \pi_j \geq w(N - S) \). Since \( S \) was an arbitrary coalition of bidders, we have that for every coalition \( T = N - S \) that includes the seller, \( \sum_{j \in T} \pi_j \geq w(T) \). Since coalitions without the seller have value zero and can therefore never block, we have shown that there is no blocking coalition. Together with efficiency, this implies that \( \pi \in \text{Core}(N, w) \).
IV. Truncation Reports and Equilibrium

In the marriage problem, a truncation report refers to a reported ranking by person \( j \) that preserves the person’s true ranking of possible partners, but which may falsely report that some partners are unacceptable. For an auction setting with transferable utility, a truncation report is similarly defined to correctly rank all pairs consisting of a non-null goods assignment and a payment but which may falsely report that some of these are unacceptable. When valuations are quasi-linear, a reported valuation is a truncation report exactly when all reported values of non-null goods assignments are reduced by the same non-negative constant. We record that observation as a lemma.

Lemma 2. A report \( \hat{u}_j \) is a truncation report if and only if there exists some \( \alpha \geq 0 \) such that for all \( x_j \in X_j \), \( \hat{u}_j(x_j) = u_j(x_j) - \alpha \).

Proof. Suppose that \( \hat{u}_j \) is a truncation report. Let \( x_j \) and \( x'_j \) be two non-null packages and suppose that the reported value of \( x_j \) is \( \hat{u}_j(x_j) = u_j(x_j) - \alpha \). Then, \( (x_j, u_j(x_j) - \alpha) \) is reportedly indifferent to \( (\emptyset, 0) \). Using the true preferences, \( (x_j, u_j(x_j) - \alpha) \) is actually indifferent to \( (x'_j, u_j(x'_j) - \alpha) \) and so must be reportedly indifferent as well: \( \hat{u}_j(x_j) - u_j(x_j) - \alpha = \hat{u}_j(x'_j) - u_j(x'_j) - \alpha \). It follows that \( u_j(x'_j) - \hat{u}_j(x'_j) = u_j(x_j) - \hat{u}_j(x_j) = \alpha \).

Conversely, suppose that there exists some \( \alpha \geq 0 \) such that for all \( x_j \in X_j \), \( \hat{u}_j(x_j) = u_j(x_j) - \alpha \). Then for any two non-null packages, the reported ranking of \( (x_j, p) \) is higher than that of \( (x'_j, p') \) if and only if \( \hat{u}(x_j) - p \geq \hat{u}(x'_j) - p' \) which holds if and only if \( u(x_j) - p \geq u(x'_j) - p' \).

We refer to the truncation report in which the reported value of all non-null outcomes is \( \hat{u}_j(x_j) = u_j(x_j) - \alpha_j \) as the “\( \alpha_j \) truncation of \( u_j \).”
In full information auction analyses since that of Bertrand (1883), auction mechanisms have often been incompletely described by the payment rule and the rule that the unique highest bid, when that exists, determines the winner. Ties often occur at Nash equilibrium, however, and the way ties are broken is traditionally chosen in a way that depends on bidders’ values and not just on their bids. For example, in a first-price auction with two bidders, both bidders make the same equilibrium bid, which is equal to the lower bidder’s value. The analysis assumes that the bidder with the higher value is favored, that is, chosen to be the winner in the event of a tie. If the high value bidder were not favored, then it would have no best reply. As Simon and Zame (1990) have explained, although breaking ties using value information prevents this from being a feasible mechanism, the practice of using this tie-breaking rule for analytical purposes is an innocent one, because, for any $\varepsilon > 0$, the selected outcome lies within $\varepsilon$ of the equilibrium outcome of any related auction game in which the allowed bids are restricted to lie on a sufficiently fine discrete grid.\(^{14}\)

In view of lemma 1, for almost all reports, assignments of goods differ among core-selecting auctions only when there is a tie; otherwise, the auction is described entirely by its payment rule. We henceforth denote the payment rule of an auction by $P(\hat{u}, x)$, to make explicit the idea that the payment may depend on the goods assignment in case of ties. For example, a first-price auction with only one good for sale is any mechanism which specifies that the winner is a bidder who has made the highest bid and the price is equal to that bid. The mechanism can have any tie-breaking rule to be used so long as (3) is satisfied. In traditional parlance, the payment rule $P$ defines an *auction*, which comprises a set of mechanisms.

**Definition.** $\hat{u}$ is an equilibrium of the auction $P$ if there is some core-selecting mechanism $(f, P)$ such that $\hat{u}$ is a Nash equilibrium of the mechanism.

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\(^{14}\) See also Reny (1999).
For any auction, consider a tie-breaking rule in which bidder \( j \) is favored. This means that in the event that there are multiple goods assignments that maximize total reported value, if there is one at which bidder \( j \) is a winner, then the rule selects such a one. When a bidder is favored, that bidder always has some best reply.

**Theorem 2.** Suppose that \((f, P)\) is a core-selecting direct auction mechanism and bidder \( j \) is favored. Let \( \hat{u}_{-j} \) be any profile of reports of bidders other than \( j \). Denote \( j \)’s actual value by \( u_j \) and let \( \pi_j = w_{\hat{u}_{-j},u_j}(N) - w_{\hat{u}_{-j},u_j}(N - j) \) be \( j \)’s corresponding Vickrey payoff. Then, the \( \pi_j \) truncation of \( u_j \) is among bidder \( j \)’s best replies in the mechanism and earns \( j \) the Vickrey payoff \( \pi_j \). Moreover, this remains a best reply even in the expanded strategy space in which bidder \( j \) is free to use shills.

**Proof.** Suppose \( j \) reports the \( \pi_j \) truncation of \( u_j \). Since the mechanism is core-selecting, it selects individually rational allocations with respect to reported values. Therefore, if bidder \( j \) is a winner, its payoff is at least zero with respect to the reported values and hence at least \( \pi_j \) with respect to its true values.

Suppose that some report \( \hat{u}_j \) results in an allocation \( \hat{x} \) and a payoff for \( j \) strictly exceeding \( \pi_j \). Then, the total payoff to the other bidders is less than \( w_{\hat{u}_{-j},u_j}(N) - \pi_j \leq w_{\hat{u}_{-j},u_j}(N - j) \), so \( N - j \) is a blocking coalition for \( \hat{x} \), contradicting the core-selection property. This argument applies also when bidder \( j \) uses shills. Hence, there is no report yielding a profit higher than \( \pi_j \), even on the extended strategy space that incorporates shills.

Since reporting the \( \pi_j \) truncation of \( u_j \) results in a zero payoff for \( j \) if it loses and non-negative payoff otherwise, it is always a best reply when \( \pi_j = 0 \).
Next, we show that the truncation report always wins for \( j \), therefore yielding a profit of at least \( \pi_j \) so that it is a best reply. Regardless of \( j \)'s reported valuation, the total reported payoff to any coalition excluding \( j \) is at most \( w_{u_j,\pi_j}(N-j) = \max_{x=(\emptyset,x_j,x_i) \in \mathcal{X}_i} \sum_{i \in N-j} \hat{u}_i(x) \). If \( j \) reports the \( \pi_j \) truncation of \( u_j \), then the maximum value is at least \( \max_{x \in \mathcal{X}_i} \left( \sum_{i \in N-j} \hat{u}_i(x) + u_j(x) \right) = w_{u_j,\pi_j}(N) - \pi_j \), which is equal to the previous sum by the definition of \( \pi_j \). Applying lemma 1 and the hypothesis that \( j \) is favored establishes that \( j \) is a winner.

**Definition.** An imputation \( \pi \) is *bidder optimal* if \( \pi \in \text{Core}(N,u) \) and there is no \( \hat{\pi} \in \text{Core}(N,u) \) such that for every bidder \( j \), \( \pi_j \leq \hat{\pi}_j \) with strict inequality for at least one bidder. (By extension, a feasible allocation is *bidder optimal* if the corresponding imputation is so.)

Next is one of the main theorems, which establishes a kind of equilibrium equivalence among the various core-selecting auctions. We emphasize, however, that the strategies require each bidder \( j \) to know the equilibrium payoff \( \pi_j \), so what is being described is a full information equilibrium but not an equilibrium in the model where each bidder’s own valuation is private information.

**Theorem 3.** For every valuation profile \( u \) and corresponding bidder optimal imputation \( \pi \), the profile of \( \pi_j \) truncations of \( u_j \) is a full information equilibrium profile of every core-selecting auction. The equilibrium goods assignment \( x^* \) maximizes the true total value \( \sum_{i \in N} u_i(x_i) \), and the equilibrium payoff vector is \( \pi \) (including \( \pi_0 \) for the seller).\(^{15}\)

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\(^{15}\) Versions of this result were derived and reported independently by Day and Raghavan (2007) and by Milgrom (2006). The latter paper was folded into Day and Milgrom (2007).
Proof. For any given core-selecting auction, we study the equilibrium of the corresponding mechanism that, whenever possible, breaks ties in (3) in favor of the goods assignment that maximizes the total value according to valuations \( u \). If there are many such goods assignments, any particular one can be fixed for the argument that follows.

First, we show that no goods assignment leads to a reported total value exceeding \( \pi_0 \).

Indeed, let \( S \) be the smallest coalition for which the maximum total reported value exceeds \( \pi_0 \).

By construction, the bidders in \( S \) must all be winners at the maximizing assignment, so

\[
\pi_0 < \max_{x_0 \in X_0} u_0(x_0) + \sum_{i \in S - 0} (u_i(x_i) - \pi_i) \leq w_u(S) - \sum_{i \in S - 0} \pi_i.
\]

This contradicts \( \pi \in Core(N, w_u) \), so the winning assignment has a reported value of at most \( \pi_0 \): \( w_u(N) \leq \pi_0 \). If \( j \) instead reports truthfully, it can increase the value of any goods allocation by at most \( \pi_j \), so

\[
w_{u, \delta_j}(N) \leq \pi_0 + \pi_j.
\]

Next, we show that for any bidder \( j \), there is some coalition excluding \( j \) for which the maximum reported value is at least \( \pi_0 \). Since \( \pi \) is bidder optimal, for any \( \varepsilon > 0 \),

\[
(\pi_0 - \varepsilon, \pi_j + \varepsilon, \pi_{-j}) \notin Core(N, w_u).
\]

So, there exists some coalition \( S' \) to block it:

\[
\sum_{i \in S'} \pi_i - \varepsilon < w_u(S').
\]

By inspection, this coalition includes the seller but not bidder \( j \). Since this is true for every \( \varepsilon \) and there are only finitely many coalitions, there is some \( S \) such that

\[
\sum_{i \in S} \pi_i \leq w_u(S). \quad \text{The reverse inequality is also implied because} \quad \pi \in Core(N, w_u), \quad \text{so}
\]

\[
\sum_{i \in S} \pi_i = w_u(S).
\]

For the specified reports, \( w_u(S) = \max_{x_0 \in X_0} \sum_{i \in S} \hat{u}_i(x_i) \geq
\]

\[
\max_{x_0 \in X_0} u_0(x_0) + \sum_{i \in S - 0} (u_i(x_i) - \pi_i) \geq w_u(S) - \sum_{i \in S - 0} \pi_i = \pi_0. \quad \text{Since the coalition value cannot decrease as the coalition expands,} \quad w_u(N - j) \geq \pi_0. \quad \text{By definition of the coalition value functions,}
\]

\[
w_u(N - j) = w_{u, \delta_j}(N - j).
\]
Using Theorem 2, j’s maximum payoff if it responds optimally and is favored is
\[ w_{u_j,\tilde{u}_j}(N) - w_{u_j,\tilde{u}_j}(N - j) \leq (\pi_0 + \pi_j) - \pi_0 = \pi_j. \]
So, to prove that the specified report profile is an equilibrium, it suffices to show that each player j earns \( \pi_j \) when these reports are made.

The reported value of the true efficient goods assignment is at least
\[ \max_{x_0} u_0(x_0) + \sum_{i \in N - 0} (u_i(x_i) - \pi_i) = w(N) - \sum_{i \in N - 0} \pi_i = \pi_0. \]
So, with the specified tie-breaking rule, if the bidders make the specified truncation reports, the selected goods assignment will maximize the true total value.

Since the auction is core-selecting, each bidder j must have a reported profit of at least zero and hence a true profit of at least \( \pi_j \), but we have already seen that these are also upper bounds on the payoff. Therefore, the reports form an equilibrium; each bidder j’s equilibrium payoff is precisely \( \pi_j \), and that the seller’s equilibrium payoff is
\[ w_s(N) - \sum_{i \in N - 0} \pi_i = \pi_0. \]

V. Minimizing Incentives to Misreport

Despite the similarities among the core-selecting mechanisms emphasized in the previous section, there are important differences among the mechanisms in terms of incentives to report valuations truthfully. For example, when there is only a single good for sale, both the first-price and second-price auctions are core-selecting mechanisms, but only the latter is strategy-proof.

To evaluate simultaneously all bidders’ incentives to deviate from truthful reporting, we introduce the following definition.

Definition. The incentive profile for a core-selecting auction \( P \) at \( u \) is
\[ e^P = \{e^P_j(u)\}_{j \in N - 0}, \]
where \( e^P_j(u) = \sup_{\tilde{u}_j} u_j(f_j(u_{-j}, \tilde{u}_j)) - P(u_{-j}, \tilde{u}_j, f_j(u_{-j}, \tilde{u}_j)) \) is j’s maximum gain from deviating from truthful reporting when \( j \) is favored.
Our idea is to minimize these incentives to deviate from truthful reporting, subject to selecting a core allocation. Since the incentives are represented by a vector, we use a Pareto-like criterion.

**Definitions.** A core-selecting auction $P$ provides suboptimal incentives at $u$ if there is some core-selecting auction $\hat{P}$ such that for every bidder $j$, $e_j^{\hat{P}}(u) \leq e_j^P(u)$ with strict inequality for some bidder. A core-selecting auction provides optimal incentives if there is no $u$ at which it provides suboptimal incentives.

**Theorem 4.** A core-selecting auction provides optimal incentives if and only if for every $u$ it chooses a bidder optimal allocation.

**Proof.** Let $P$ be a core-selecting auction, $u$ a value profile, and $\pi$ the corresponding auction payoff vector. From theorem 2, the maximum payoff to $j$ upon a deviation is $\pi_j$, so the maximum gain to deviation is $\pi_j - \pi_j$. So, the auction is suboptimal exactly when there is another core-selecting auction with higher payoffs for all bidders, contradicting the assumption that $\pi$ is bidder optimal.

Recall that when the Vickrey outcome is a core allocation, it is the unique bidder optimal allocation. So, Theorem 4 implies that any core-selecting auction that provides optimal incentives selects the Vickrey outcome whenever that outcome is in the core with respect to the reported preferences. Moreover, because truthful reporting then provides the bidders with their Vickrey payoffs, theorem 2 implies the following.

**Corollary.** When the Vickrey outcome is a core allocation, then truthful reporting is an ex post equilibrium for any mechanism that always selects bidder optimal core

Among the bidder-optimal core-selecting auctions, one particularly interesting set is the class of minimum-revenue core-selecting auctions.
**Definition.** A core-selecting auction $P(u,x)$ is a *minimum-revenue core-selecting auction* if there is no other core-selecting auction $\hat{P}(u,x)$ such that $\sum_{j \in \hat{J}} \hat{p}_j < \sum_{j \in J} p_j$.

Since the allocation $x$ does not vary among core-selecting auctions, it is obvious from the defining inequality that no other core-selecting auction can lead to a higher payoff (and hence a lower price) for each bidder.

**Lemma 3.** Every minimum-revenue core-selecting auction $P(u,x)$ is bidder-optimal.

The converse of Lemma 3 is not true in general. As a counterexample, let suppose there are five bidders: $J = 5$.\textsuperscript{16} Let each feasible $X_j$ be a singleton; each bidder is interested in only one package, a condition often called *single-minded* bidding. Further, let $u_j(x_j) = 2$, for all $j$, and let $x_1, x_2, x_3$, be mutually disjoint, while $x_4 = x_1 \cup x_2$ and $x_5 = x_3$. For example, bidders could be interested in items from the set $\{A, B, C\}$ with bundles of interest $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, and $\{B, C\}$, respectively. For these parameters, bidders 1, 2, and 3 win their bundles of interest in the unique efficient allocation. But a valid bidder-optimal rule may select payments $(1, 1, 1)$ with total revenue of 3, while the unique minimum-revenue solution is $(0, 2, 0)$, confirming that not all bidder-optimal payment rules minimize revenue within the core. To see that $(1, 1, 1)$ is indeed bidder-optimal, note that any single or joint reduction in payment from that point will induce a blocking coalition involving one or the other of the losing bidders.

Since minimum-revenue core-selecting auctions are bidder-optimal, they inherit the properties of that larger class. The next theorem asserts that minimum-revenue core-selecting auctions have an additional optimality property.

\textsuperscript{16} Our counter-example has three winning bidders. There are no counter-examples with fewer than three winners.
Theorem 5. If $\hat{P}$ is a minimum-revenue core-selecting auction, then for any fixed $u$ and corresponding efficient allocation $x$:

$$\hat{P}(u,x) \in \arg \min \sum_{j=1}^{n} \epsilon_j^{\hat{P}}(u)$$

Proof. Again from Theorem 2, we have a maximum possible gain from deviation given by

$$\epsilon_j^{\hat{P}}(u) = \pi_j - \pi_j$$

for each bidder, which given any fixed value-maximizing $x$ is equal to $P_j - \bar{P}_j$. Thus, $\arg \min \sum_{j=1}^{n} \epsilon_j^{\hat{P}}(u) = \arg \min \sum_{j=1}^{n} (P_j - \bar{P}_j) = \arg \min \sum_{j=1}^{n} P_j$, with the second equality following since $\bar{P}_j$ is a constant with respect to $P$, and the main result following by the revenue minimality of $\hat{P}$.

VI. Connections to the Marriage Problem

Even though Theorems 2-4 in this paper are proved using transferable utility and do not extend to the case of budget-constrained bidders, they do all have analogs in the non-transferable utility marriage problem.

Consider Theorem 2. Roth and Peranson (1999) have shown for a particular algorithm in the marriage problem that any fully informed player can guarantee its best stable match by a suitable truncation report. That report states that all mates less preferred than its best achievable mate are unacceptable. The proof in the original paper makes it clear that their result extends to any stable matching mechanism, that is, any mechanism that always selects a stable match.

Here, in correspondence to stable matching mechanisms, we study core-selecting auctions. For the auction problem, Ausubel and Milgrom (2002) showed that the best payoff for any bidder at any core allocation is its Vickrey payoff. So, the Vickrey payoff corresponds to the best mate assigned at any stable match. Thus, the auction and matching procedures are connected not just by the use of truncation strategies as best replies but by the point of the truncation, which is at the player’s best core or stable outcome.
Theorem 3 concerns Nash equilibrium. Again, the known results of matching theory are similar. Suppose the participants in the match in some set $S^C$ play non-strategically, like the seller in the auction model, while the participants in the complementary set $S$, whom we shall call bidders, play Nash equilibrium. Then, for bidder-optimal stable match, the profile at which each player in $S$ reports that inferior matches are unacceptable is a full-information Nash equilibrium profile of every stable matching mechanism and it leads to that $S$-optimal stable match. This result is usually stated using only men or women as the set $S$, but extending to other sets of bidders using the notion of bidder optimality is entirely straightforward.

Finally, for Theorem 4, suppose again that some players are non-strategic and that only the players in $S$ report strategically. Then, if the stable matching mechanism selects an $S$-optimal stable match, then there is no other stable matching mechanism that weakly improves the incentives of all players to report truthfully, with strict improvement for some. Again, this is usually stated only for the case where $S$ is the set of men or the set of women, and the extension does require introducing the notion of a bidder optimal match.

**VII. Corrections and Other Related Literature**

The original paper on which this chapter was based (Day and Milgrom 2007) claimed an additional theorem about revenue monotonicity of the minimum-revenue core-selecting auction, namely, that the seller’s revenue weakly increases as bid values increase or alternatively as additional bidders enter the auction. This claim later proved to be erroneous. This error was brought to our attention in independent contributions by Ott (2009) and Lamy (2009). Beck and Ott (2010) give necessary and sufficient conditions to characterize revenue-monotonic core-selecting auctions and find the ones with the best incentives in that set.

To illustrate the failure of revenue monotonicity in revenue-minimizing core-selecting mechanism, consider the following simple example. Let bidders 1, 2, and 3 each bid $2 on a

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17 This is defined analogously to the bidder optimal allocation.
single item of interest (say A, B and C, respectively) and let bidder 4 bid $3 on \( \{A, B\} \) while bidder 5 bids $3 on \( \{B, C\} \). Bidders 1, 2, and 3 win in the efficient allocation, while the presence of losing bidders 4 and 5 dictate core constraints on the winning bidders’ payments as follows: bidders 1 and 2 must pay at least $3 in total, and bidders 2 and 3 must pay at least $3 in total. The unique minimum revenue solution is for bidders 1, 2, and 3 to pay $1, $2, and $1, respectively. But if bidder 2 were to increase her bid to $3, the unique set of payments becomes $0, $3, $0, and the seller’s revenue has dropped from $5 to $3 following a $1 bid increase by bidder 2. Intuitively, though bidder 2’s payments count only once from the perspective of the seller, they help to satisfy two core constraints at once, in contrast to the payments of bidders 1 and 3. If we consider further bid increases by bidder 2, we see that she need not pay any more than $3, illustrating eventual revenue invariance under increases in a truncation strategy – a property first described by Day and Cramton (2010).

Despite the non-monotonicity of some core-selecting auctions, this class continues to be studied and applied in practice. Goeree and Lien (2009) demonstrate a revenue weaknesses of core-selecting auctions under Bayes-Nash equilibrium in a limited setting, while related work of Rastegari, Condon and Leyton-Brown (2010) provides impossibility results for revenue monotonicity under a variety of assumptions. In a more positive stream, Erdil and Klemperer (2009) introduce refined rules for core-selecting auctions to mitigate incentives for small deviations (as opposed to maximal incentives to deviate treated in Theorems 4 and 5.) Some the strongest support for core-selecting auctions among the more recent literature is given by Othman and Sandholm (2010), who introduce envy-reduction auction protocols that result in core outcomes. Day and Cramton (2010) also demonstrate an envy-reduction result, that truncation strategies result in envy-free outcomes in core-selecting auctions.
VIII. Conclusion

We motivated our study of core-selecting auctions both by their practical interest and by their relations to stable matching mechanisms. The evidence from case studies and from the Kagel-Roth laboratory experiments, which shows that participants are quick to stop using certain unstable matching mechanisms but that stable mechanisms persist, has usually been understood to be applicable in general to matching mechanisms. But there is no obvious reason to accept that as the relevant class. The usual theoretical arguments about the continued use of a mechanism distinguish core-selecting mechanisms from other mechanisms. That applies equally for auctions and matching problems, and the failure to reject the narrower theoretical hypothesis is also a failure to reject the broader one.

Despite the theoretical similarities between auction and matching mechanism, stable matching mechanisms for multi-item applications have so far been more extensively used in practice. It is possible that this is about to change. The two complexity challenges that are posed by core-selecting auctions – computational complexity and communications complexity – are both being addressed in research and in practice.

The computations required by core-selecting auctions are, in general, much harder than those for matching and computational tractability for problems of interesting scale has only recently been achieved. Indeed, Day and Raghavan (2007) showed that the computational complexity of finding core outcomes is equivalent to the complexity of the corresponding efficient allocation problem, and is thus NP-hard in the most general case. The implementation of core-selecting auctions is limited primarily by our ability to solve larger and larger NP-hard problems, or to find reasonable application-specific restrictions on bidding that make the problem tractable. And, efforts are being made to find just such restrictions. For example, the core-selecting European spectrum auctions to date have each described their sets of objects in ways
that made for comfortably small optimization problems, which can be solved relatively quickly on a desktop computer.

The issue of communications complexity can be highlighted with some simple arithmetic. In an environment with \( N \) items for sale, the number of non-empty packages for which a bidder must report values is \( 2^N - 1 \). That is unrealistically large for most applications if \( N \) is even a small two-digit number. For the general case, Segal (2003) has shown that communications cannot be much reduced without severely limiting the efficiency of the result.

But communication complexity need not definitively rule out core-selecting package auctions. In many real-world settings, the auctioneer can simplify the problem by limiting the packages that can be acquired or by engaging in conflation, according to which similar items are treated as if they were identical (Milgrom (2010)). An auctioneer may know that radio spectrum bands must be compatible with international standards, or that complementarities in electrical generating result from costs saved by operating continuously in time, minimizing time lost when the plant is ramped up or down, or that a collection of airport landing rights between 2:00-2:05 can be conflated without much loss with rights between 2:05-2:10 and 2:10-2:15. And for some classes of preferences, such as the case where goods are substitutes, substantial progress on compact expressions of values has already been made.\(^{18}\) Practical designs that take advantage of such knowledge can still be core-selecting mechanisms and yet can entail compact reporting by bidders.

The class of core-selecting auctions includes the pay-as-bid “menu auction” design studied by Bernheim and Whinston (1986), the ascending proxy auction studied by Ausubel and Milgrom (2002) and Parkes and Ungar (2000), the assignment auction introduced in Milgrom (2009a, b), and any of the mechanisms resulting from the core computations in Day and Raghavan (2007),

\(^{18}\) Hatfield and Milgrom (2005) introduced the endowed assignment valuations for this purpose.
Day and Cramton (2010), or Erdil and Klemperer (2009). Several of these are the very minimum-revenue core-selecting auctions that continue to be proposed for high-stakes applications.
References


Ott, Marion (2009). *Second-Price Proxy Auctions in Bidder-Seller Networks.*


