Investment Incentives in Near-Optimal Mechanisms*

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February 25, 2020

Abstract

In a Vickrey auction, if one bidder has an option to invest to increase his value, the combined mechanism including investments is still fully optimal. In contrast, for any $\beta < 1$, we find that there exist monotone allocation rules that guarantee a fraction $\beta$ of the allocative optimum in the worst case but such that the associated mechanism with investments by one bidder can lead to arbitrarily small fractions of the full optimum being achieved. We show that if a monotone allocation rule satisfies a new property called ARNIE and guarantees a fraction $\beta$ of the allocative optimum, then in the equilibrium of the threshold auction game with investments, at least a fraction $\beta$ of the full optimum is achieved. We also establish generalizations and a partial converse, and show that some well-known approximation algorithms satisfy the ARNIE property.

Keywords: Combinatorial optimization, Knapsack problem, Investment, Auctions, Approximation, Algorithms

JEL classification codes: D44, D47, D82

*We thank Andy Haupt, Zi Yang Kang, Ellen Muir, Noam Nisan, Amin Saberi, and Mitchell Watt for helpful comments. We thank Broadsheet Cafe for inspiration and coffee. Akbarpour and Kominers gratefully acknowledge the support of the Washington Center for Equitable Growth. Additionally, Kominers gratefully acknowledges the support of National Science Foundation grant SES-1459912 and both the Ng Fund and the Mathematics in Economics Research Fund of the Harvard Center of Mathematical Sciences and Applications. All errors remain our own.

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Electronic copy available at: https://ssrn.com/abstract=3544100
1 Introduction

Recent research in economics and computation has focused on identifying fast algorithms for finding approximate solutions to computationally hard allocation problems—along with associated incentive compatible payment schemes. A popular example is the combinatorial auction, in which computing an optimum is often NP-hard. Yet analyses that focus just on the difficult allocation problem and its implementation using prices are what economists think of as short-run analyses: they take the resource constraints as given, omitting the long-run consideration of how parties might invest to create new assets or improve existing ones or disinvest to cash in less valuable assets.

In resource allocation problems, investment choices can sometimes affect both what is feasible (such as when an airline that chooses to use larger planes is more difficult to schedule on a runway) and the values of the items being allocated (because a larger plane carries more passengers). If the participants’ investment payoffs were well aligned with the total welfare objective, then they would tend to invest or disinvest in ways that improve the long-run performance of the whole system. In approximate mechanisms, however, payoffs are often poorly aligned.

Privately profitable investments can reduce welfare in mechanisms based on popular fast algorithms. To illustrate this, consider a simple knapsack problem in which each item that might be packed has a different owner. Suppose that the owners are competing to buy space in the knapsack and that the auctioneer knows the sizes of the items but not their values, so the auctioneer uses the owners’ bids to determine which items are packed.\(^1\) Since the knapsack problem is NP-hard, the auctioneer applies a fast algorithm—in this example, Dantzig’s greedy algorithm—to the bids and sizes to determine the winner. This algorithm sorts items in decreasing order of value-per-unit-size, and packs items in that order, stopping when it encounters an item that does not fit. By setting the price to be paid for any packed item equal to the item’s “threshold value,” the result is a truthful auction, that is, each bidder’s optimal strategy is to bid his value, regardless of how others may bid.\(^2\)

Here is a simple numerical example to show that private investment incentives in the threshold auction based on the greedy algorithm can be damaging to welfare. The knapsack has capacity 20 and three bidders have values of 11, 11, and 12 and items of sizes 10, 10, and 11. In the threshold auction, bidders bid their values truthfully and those values are used

\(^{1}\)This has some economic applications. If a seller has a fixed quantity of some resource — space on a cargo ship, time on a cloud computer — and supplying each buyer takes a known quantity of that resource, then the seller faces a knapsack problem.

\(^{2}\)The threshold value is defined to be the lowest value the bidder could bid to be packed by the greedy algorithm, taking all the other owners’ bids as given. The truthfulness of this mechanism and the ease of computing threshold prices for it were established by Lehmann et al. (2002).
by the greedy algorithm. Since $\frac{11}{10} > \frac{12}{11}$, the greedy algorithm packs the first two items for a total value of 22, which is also the optimum for that problem. Now we add an investment stage. Suppose that before the auction, the third bidder has an opportunity to increase his value from 12 to 14 at a cost of 1. From the bidder’s perspective, the investment can be assessed like this: “If I invest, my value will be 14 and my item will be packed. In fact, any value over 12.1 would result in my item being packed ($\frac{11}{10} = \frac{12.1}{11}$), so 12.1 is my threshold price. If I invest, I will pay that threshold price of 12.1 plus my investment cost of 1, but my total cost of 13.1 is less than my value of 14 for a place in the knapsack. That’s a good deal! I should invest.” From a social welfare perspective, the investment is assessed differently. If the bidder invests, the packed value will be 14 and an investment cost of 1 will be incurred, for a welfare of just 13. With no investment, welfare would be 22, so the investment reduces welfare.

In this paper, we study a long-run formulation in which the resource allocation mechanism consists of two stages, beginning with an investment stage in which one bidder can make a costly investment. The first-stage investment determines his value, which is used by the second-stage algorithm to compute the final allocation. We limit attention to algorithms that satisfy weak monotonicity, which are those that can be truthfully implemented by some auction mechanism (Nisan, 2000; Saks and Yu, 2005). Our central question is this: if an approximate algorithm delivers at least a fraction $\beta \in (0, 1)$ of the optimal welfare in all instances of the allocation problem, when does the same guarantee $\beta$ apply, for all investment technologies, to the two-stage mechanism?

As a benchmark, consider participants’ investment incentives when the algorithm is based on exact optimization. In that case, the truthful mechanism is the Vickrey-Clarke-Groves (VCG) auction, which provides each participant with an incentive to report his values truthfully (Green and Laffont, 1977; Holmström, 1979). In a VCG auction, each participant’s equilibrium payoff is equal to his contribution to total welfare. Consequently, for any investment cost function mapping bidder values into costs, the bidder’s payoff-maximizing investment decision is the same as the one that maximizes total welfare.

How much of this benchmark analysis extends to other truthful mechanisms? In any truthful mechanism, the price a participant must pay to achieve his outcome depends only on other participants’ values, so holding the allocation fixed, any extra value that results when the value of a bidder’s outcome is increased adds an equal amount to the bidder’s payoff. That aligns one term of the investment incentive with social welfare in roughly the same way as the VCG auction. In some truthful mechanisms, however, there is an additional term in social welfare computation: an individually profitable investment decision that leaves the participant’s outcome unchanged may cause an externality, changing the other participants’
outcomes in a way that affects their welfare. There is no such externality in a VCG auction, because when a participant’s value is changed in a way that does not affect his outcome, the overall allocation also remains unchanged.

We report four main findings to characterize how investments affect worst-case performance in the long run. First, we show that some algorithms with good worst-case guarantees for the short-run problem have arbitrarily bad guarantees for the long-run problem. More precisely, for any $\beta < 1$, there is an algorithm for the knapsack problem that guarantees at least a fraction $\beta$ of the optimum in the short-run problem, but for any $\varepsilon > 0$, that same algorithm guarantees no more than a fraction $\varepsilon$ in the long-run problem.

Second, we identify a property that we call ARNIE (“avoiding relevant negative investment externalities”) such that, for every problem and every investment cost function, the long-run, worst-case performance guarantee of an ARNIE algorithm is exactly the same as the short-run, worst-case guarantee. Given an allocation problem, one can restrict attention to sub-problems to impose more structure; doing so weakly improves an algorithm’s short-run guarantee. An algorithm has the same short-run and long-run guarantees on every sub-problem if it is ARNIE.

The definition of ARNIE is as follows: Given any value profile and feasibility constraints, an algorithm outputs some set of packed bidders. Suppose we raise the value of a packed bidder, or lower the value of an unpacked bidder, and then run the algorithm at the new value profile. The algorithm is ARNIE if the new packing, assessed at the new values, yields at least as much welfare as the old packing, assessed at the new values.

In the knapsack problem, the greedy algorithm generally guarantees only 0, but for the sub-problem with knapsack capacity 20 and item sizes 10, 10 and 11, the short-run guarantee is $\frac{11}{20}$ of the optimum. We will show later that the greedy algorithm is ARNIE, so our theorem implies that the long-run sub-problem with any investment technology satisfies the same $\frac{11}{20}$ guarantee.

Our third result is a characterization: an algorithm has the same short-run and long-run guarantees on every sub-problem if and only if it satisfies a slightly weakened version of ARNIE. The weakened version of ARNIE, however, is much more difficult to verify than simple ARNIE.

Finally, we identify some interesting algorithms that satisfy ARNIE. Among them is a canonical Fully Poly-Time Approximation Scheme (FPTAS) for the knapsack problem, which for any $\varepsilon > 0$, guarantees a value within $(1 - \varepsilon)$ times the maximum.

The ARNIE acronym introduces the concept of “relevant” externalities, which in this paper refers to externalities that may, for some cost function, cause the long-run problem to violate the worst-case performance guarantee of the short-run problem. Some externalities
cannot cause such a violation. For example, in the knapsack problem described above, although bidder 3’s investment caused the performance ratio to fall far below 100%, the short-run problem without investment performs even worse when bidder 3’s value is set slightly above his threshold price of 12.1. For our analysis, we will require the set of possible values for a bidder to be an interval, so if a bidder’s investment can change it from loser to winner, then it is possible to have a value near the threshold price.

Let us generalize that example to highlight the externalities that cannot be relevant. For any monotone algorithm and associated threshold auction, if an investment is profitable for a bidder and raises his value to just above his threshold, the investment must cost approximately zero. Consequently, the long-run performance for that problem must be nearly identical to that of the related short-run problem in which the bidder’s value is equal to his threshold price. Even if, as in the example, the investment that raises value up to the threshold also reduces performance, that is nevertheless just the same as the short-run performance for the related problem, so the investment does not degrade the performance guarantee. Such an externality is not relevant.

In contrast, consider a problem in which the bidder’s value is already above the threshold and there is an investment that increases the bidder’s value by some amount \( \Delta \). For a worst-case analysis, suppose the cost is also (nearly) \( \Delta \). Such an investment adds (nearly) nothing to the \textit{optimal} value of the long-run problem, because any extra return is exactly offset by the extra cost. For the same reason, using the algorithm and its associated threshold auction, the increase in value for the bidder is exactly offset by the investment cost, so the effect of the investment on total welfare is equal to the change in the value of the allocation to other bidders, which is exactly the externality caused by the investment. If that externality is negative, then the long-run problem with investment has a strictly worse performance ratio than the short-run problem without investment. If the short-run problem is close to the worst-case or if the negative externality is large, then the performance ratio for the long-run problem will fall below the short-run guarantee. ARNIE avoids such possibilities by ruling out this kind of relevant negative externality.

If the bidder initially had a value below the threshold, then the effect of a profitable investment can be decomposed into two steps: first increase the value to the threshold value at zero cost and then increase it from there to his final value at the full cost. The preceding analysis shows that the first step cannot cause a violation of the performance guarantee and ARNIE ensures that the second step cannot cause a violation either.

Similarly, a disinvestment from above the threshold value to below can also be decomposed into two steps: to the just below threshold value and then to a lower value. The disinvestment that reduces the bidder’s value to just below the threshold cannot reduce
the performance ratio below short-run performance in that problem. ARNIE rules out the possibility that there is any negative externality from reducing the value further.

The simple message is this: all negative externalities in an algorithm can reduce welfare given some investment cost function, but only the two kinds of externalities highlighted by ARNIE can reduce the algorithm’s long-run performance below the short-run guarantee. By ruling out these externalities, ARNIE ensures that the long-run guarantee is exactly equal to the short-run guarantee. The analysis extends to bidders with multi-dimensional values, for an appropriate generalization of ARNIE.

In addition to the four general findings, we report two others. One concerns the equilibrium in investments when several bidders may invest. Even in a Vickrey auction, a Nash equilibrium can have inefficient investments due to a coordination failure among bidders, but for the Vickrey auction there is also a Nash equilibrium in which the investments preserve the efficiency of the mechanism. We show that if a monotone algorithm is non-bossy, which is more restrictive than ARNIE, and guarantees a $\beta$ fraction of the optimum for all instances of the short-run problem, then there exists an equilibrium of the long-run problem enjoys the same $\beta$ guarantee. The second concerns combinatorial auctions in which the set of values is restricted for tractability to be fractionally subadditive. For that case, we show that if the investment cost function is isotone and supermodular, then for any ARNIE algorithm, the long-run performance guarantee is again the same as the short-run performance guarantee.

\section{1.1 Related work}

Economists have studied \textit{ex ante} investment in mechanism design at least since the work of Rogerson (1992), who demonstrated that Vickrey mechanisms induce efficient investment. Bergemann and Välimäki (2002) echoed and extended this finding in a setting with uncertainty, in which agents invest in information before participating in an auction. Relatedly, Arozamena and Cantillon (2004), studied pre-market investment in procurement auctions, showing that while second-price auctions induce efficient investment, first-price auctions do not. Hatfield et al. (2014, 2019) extended these findings to characterize a relationship between the degree to which a mechanism fails to be strategy-proof and/or efficient and the degree to which it fails to induce efficient investment. While that paper, like ours, deals with the connection between (near-)efficiency at the allocation stage and (near-)efficiency at the investment stage, it uses additive error bounds, rather than the multiplicative worst-case bounds that are standard for the analysis of computationally hard problems.

Our paper is also not the first work to study investment incentives in an NP-hard allocation setting. Milgrom (2017) introduced a “knapsack problem with investment” in which the
items to be packed are owned by individuals, and owners may invest to make their items either more valuable or smaller (and thus easier to fit into the knapsack). In the present paper, we reformulate the investment question in terms of worst-case guarantees and broaden the formulation to study incentive-compatible mechanisms for a wide class of resource allocation problems.

Lipsey and Lancaster (1956) explain that in economic systems that are not fully optimized, investments that violate optimality conditions can sometimes improve welfare by offsetting other shortcomings of the system. Our focus on worst-case performance and a particular definition of negative externalities leads to a more definite conclusion: ruling out the “relevant” negative externalities highlighted by ARNIE ensures that the long-run performance is the same as the short-run guarantee.

By studying the investment problem in near-optimal mechanisms, our paper is naturally connected to a large literature, primarily in computer science, that considers computational complexity in mechanism design, and explores properties of approximately optimal mechanisms. Among these works are those of Nisan and Ronen (2007) and Lehmann et al. (2002). Nisan and Ronen (2007) showed that in settings where identifying the optimal allocation is an NP-hard problem, VCG-based mechanisms with nearly optimal allocations determined by heuristics are generically non-truthful, while Lehmann et al. (2002) introduced a truthful mechanism for the knapsack problem in which the allocation is determined by a greedy algorithm. In addition, Hartline and Lucier (2015) developed a method for converting a (non-optimal) algorithm for optimization into a Bayesian incentive compatible mechanism with weakly higher social welfare or revenue; Dughmi et al. (2017) generalized this result to multidimensional types.

There is also a large literature on greedy algorithms of the type we study here, which sort bidders based on some intuitive criteria and choose them for packing in an irreversible way; see Pardalos et al. (2013) for a review. Lehmann et al. (2002) study the problem of constructing strategy-proof mechanisms from greedy algorithms; similarly, Bikhchandani et al. (2011) and Milgrom and Segal (2020) propose clock auction implementations of greedy allocation algorithms.

Our concept of an ARNIE algorithm is closely related to the definition of a ‘bitonic’ algorithm, introduced by Mu’Alem and Nisan (2008) to construct truthful mechanisms in combinatorial auctions. Bitonicity is defined for binary outcomes; with the restriction to binary outcomes, every ARNIE algorithm is bitonic, but not vice versa.
2 Investment with binary outcomes

2.1 Model

We start our exposition with binary outcomes—each bidder is either ‘packed’ or ‘unpacked’, and we normalize the value of being unpacked to 0. We later generalize the main theorem to finitely many outcomes.

We consider three nested perspectives on the same situation. First, the allocation problem, in which our objective is total welfare and the values of the bidders are known to us. Second, the reporting problem, in which values are private information and we must elicit them via an incentive-compatible payment rule prior to allocation. Third, the investment problem, in which a bidder can make costly investments to change his value before reporting.

2.1.1 The allocation problem

We start by defining allocation problems. In an allocation problem, we directly observe each bidder’s value and choose a set of bidders to pack, subject to feasibility constraints. Our objective is the sum of the values of the packed bidders.

An instance \((v, A)\) consists of:

1. a value profile \(v \in (\mathbb{R}_0^+)^N\), for some set of bidders \(N\), and
2. a set of feasible allocations \(A \subseteq \wp(N)\).

An allocation problem is a collection of instances, denoted \(\Omega\). We assume that the possible value profiles are products of intervals, that is, for each \(A\) and each \(n \in N\), there is some interval \(V^A_n \subseteq \mathbb{R}\) such that \(\{v : (v, A) \in \Omega\} = \prod_n V^A_n\).

An algorithm \(x\) selects, for each instance \((v, A) \in \Omega\), a feasible allocation, that is, \(x(v, A) \in A\).\(^3\) We will occasionally abuse notation and write \(x_n(v, A)\) to denote an indicator function, equal to 1 if \(n \in x(v, A)\) and 0 otherwise.

The welfare of algorithm \(x\) at instance \((v, A)\) is

\[
W_x(v, A) \equiv \sum_{n \in x(v, A)} v_n.
\]

\(^3\)In complexity theory, we often are not given the feasible allocations \(A\) directly, but instead only a description that implies which allocations are feasible. For instance, a description could specify the items’ sizes and the capacity of the knapsack. In principle, algorithms for the knapsack problem could output different allocations for two instances with different item sizes but the same feasible allocations. Our formulation ignores this description-dependence, but we could easily accommodate it by specifying a function \(A\) from descriptions to feasible allocations, and defining an instance as consisting of a value profile \(v\) and a description \(d\); none of our results would materially change with this adjustment.

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The optimal welfare at instance \((v, A)\) is

\[
W^*(v, A) \equiv W_{\text{OPT}}(v, A) = \max_{a \in A} \left\{ \sum_{n \in a} v_n \right\},
\]

where OPT is an algorithm that always achieves the maximum feasible welfare,

\[
\text{OPT}(v, A) \in \arg\max_{a \in A} \left\{ \sum_{n \in a} v_n \right\}.
\]

In the Knapsack problem and other cases of interest, optimization is NP-hard and it may be impractical to identify optimal solutions, even though fast algorithms can guarantee acceptable performance on some problems. A common measure of performance is to look for worst-case guarantees, as follows.

**Definition 2.1.** For \(\beta \in [0, 1]\), an algorithm \(x\) is a \(\beta\)-approximation for allocation if for all \((v, A) \in \Omega\)

\[
\beta W^*(v, A) \leq W_x(v, A).
\]

Our goal is to characterize when the worst-case approximation performance of an algorithm can be extended to the same approximation even when we include the incentives of the asset owners to invest in their assets and to report their values truthfully to the mechanism. We begin with the problem of truthful reporting, which is equivalently characterized as a problem of mechanism design.

### 2.1.2 The reporting problem

Given some allocation problem \(\Omega\), we now consider the corresponding reporting problem. In the reporting problem, we no longer directly observe each bidder’s value. Instead, each bidder’s value is private information, and we must elicit that information by choosing not just an algorithm, but also a payment rule.

In the reporting problem, for each instance \((v, A)\), we face a set of bidders \(N\), each of whom has a value \(v_n\) for being packed. Now we choose not only an algorithm \(x(v, A) \in A\), but also a payment rule \(p(v, A) \in \mathbb{R}^N\). A mechanism is a pair \((x, p)\).

The feasible allocations \(A\) are common knowledge. The values \(v_n\) are private information, and each bidder reports a value \(\hat{v}_n \in V\) to the mechanism.

**Definition 2.2.** The mechanism \((x, p)\) is strategy-proof if for all \((v, A) \in \Omega\) and all \(n \in N\), we have

\[
v_n \in \arg\max_{\hat{v}_n \in I} \{v_n x_n(\hat{v}_n, v_{-n}, A) - p_n(\hat{v}_n, v_{-n}, A)\};
\]
that is, if reporting truthfully is always a best response (for each \( n \in N \)).

In the reporting problem, we choose the mechanism \((x, p)\) to (approximately) maximize welfare, subject to the additional constraint that \((x, p)\) be strategy-proof.

**Definition 2.3.** For \( \beta \in [0, 1] \), \((x, p)\) is a \( \beta \)-approximation for reporting if \( x \) is a \( \beta \)-approximation for allocation and \((x, p)\) is strategy-proof.

Given an algorithm \( x \) that is an \( \beta \)-approximation for allocation, when can we choose payments so that \((x, p)\) is an \( \beta \)-approximation for reporting?

**Definition 2.4.** Algorithm \( x \) is monotone (on \( \Omega \)) if, for all \((v, A) \in \Omega \) and \( n \in N \), if \( n \in x(v, A) \), then \( n \in x(\tilde{v}_n, v_n, A) \) for all \( \tilde{v}_n \geq v_n \).

**Definition 2.5.** The threshold price for bidder \( n \) at instance \((v, A)\) is

\[
\tau_n^x(v, A) \equiv \inf \{ \tilde{v}_n : n \in x(\tilde{v}_n, v_n, A) \text{ and } (\tilde{v}_n, v_n, A) \in \Omega \}.
\]

For any \( x \), we define the threshold auction \((x, p^x)\) to be the mechanism such that for all \( n \) and all \((v, A)\),

\[
p_n^x(v, A) = x_n(v, A) \tau_n^x(v, A);
\]

that is, a threshold auction uses the threshold allocation rule, and charges each bidder his threshold price in the case that he is packed, and charges 0 otherwise.

For any optimal algorithm OPT, the mechanism \((\text{OPT}, p^{\text{OPT}})\) is a Vickrey-Clarke-Groves (VCG) auction. For other strategy-proof mechanisms, the following characterization is a special case of the well-known “taxation principle” of mechanism design. (Alternatively, see Myerson (1981).)

**Proposition 2.1.** If \( x \) is monotone, then the threshold auction \((x, p^x)\) is strategy-proof. Conversely, if \((x, p)\) is strategy-proof and we have

\[
x_n(v, A) = 0 \implies p_n(v, A) = 0,
\]

then \( x \) is monotone and \((x, p)\) is a threshold auction.

**Corollary 2.1.** If \( x \) is monotone and a \( \beta \)-approximation for allocation, then \((x, p^x)\) is a \( \beta \)-approximation for reporting.
2.1.3 The investment problem

Given some allocation problem \( \Omega \), we now define the corresponding investment problem. The investment problem we consider can be interpreted as a long-run analysis, which complements the short-run analysis of reporting problems. In an investment problem, one bidder has an opportunity to change his value at a cost, and he does so with full information about the mechanism and about other bidders’ values.

Formally, before the auction commences, one bidder \( i \in N \) can invest to change his value. An investment is a pair \((v_i, c_i) \in V^A_i \times \mathbb{R}\), specifying a value and a cost. An instance of the investment problem is a tuple \((I_i, v_{-i}, A)\), where \( I_i \subseteq V^A_i \times \mathbb{R} \) is a set of feasible investments and \( v_{-i} \in V^A_{-i} \). We restrict attention to instances that satisfy:

1. **Finite.** \(|I_i| < \infty\).

2. **Normalization.** \( \min \{c_i : (v_i, c_i) \in I_i\} = 0 \).

Note that while \( n \) denotes a representative element of \( N \), \( i \) denotes the investor, so \( i \) is only well-defined once we fix an instance of the investment problem.

We begin by studying investments in the case that the auction is a VCG auction. For VCG auctions, the total profits of the auctioneer and all the participants besides \( i \) is an amount \( f(v_{-i}) \) that does not depend on \( i \)’s report. Hence, \( i \)’s net profit is the total social welfare minus \( f(v_{-i}) \). A consequence is that \( i \) maximizes his own payoff by maximizing social welfare, which he does both by reporting truthfully and by choosing the social-welfare maximizing investment.

**Proposition 2.2.** In the investment problem for a VCG auction, \( i \)’s payoff-maximizing investment choice also maximizes the social welfare.

Next, suppose we have some other monotone algorithm \( x \) that guarantees a \( \beta \)-approximation for allocation. Under what conditions does the corresponding threshold auction still yield a \( \beta \)-approximation in the investment problem?

When \( i \) faces a threshold auction \((x, p^x)\), his utility from investment \((v_i, c_i)\) is

\[
u_i(x, v_i, c_i, v_{-i}, A) \equiv v_i x_i(v_i, v_{-i}, A) - p^x_i(v_i, v_{-i}, A) - c_i.
\]

We denote his best responses at instance \((I_i, v_{-i}, A)\) by

\[
BR(x, I_i, v_{-i}, A) \equiv \arg \max_{(v_i, c_i) \in I_i} \{u_i(x, v_i, c_i, v_{-i}, A)\}.
\]
The welfare of algorithm \( x \) at instance \((I_\iota, v_{-\iota}, A)\) is then
\[
W_x(I_\iota, v_{-\iota}, A) \equiv \min_{(v, c) \in BR(x, I_\iota, v_{-\iota}, A)} \{W_x(v_\iota, v_{-\iota}, A) - c_\iota\}; (1)
\]
the optimal welfare at instance \((I_\iota, v_{-\iota}, A)\) is
\[
W^*(I_\iota, v_{-\iota}, A) \equiv \max_{(v, c) \in I_\iota} \{W^*(v_\iota, v_{-\iota}, A) - c_\iota\}.
\]

Definition 2.6. For \( \beta \in [0, 1] \), algorithm \( x \) is a \( \beta \)-approximation for investment if for all investment instances \((I_\iota, v_{-\iota}, A)\),
\[
\beta W^*(I_\iota, v_{-\iota}, A) \leq W_x(I_\iota, v_{-\iota}, A).
\]

Proposition 2.3. If \( x \) is a \( \beta \)-approximation for investment, then \( x \) is a \( \beta \)-approximation for allocation.

Proof. Any instance of the allocation problem \((v_\iota, v_{-\iota}, A)\) is equivalent to the instance of the investment problem \((I_\iota, v_{-\iota}, A)\) in which the investment technology is the singleton \(\{(v_\iota, 0)\}\). Thus, the investment problem embeds the allocation problem without investment as a special case. \( \Box \)

If \( x \) is a \( \beta \)-approximation for allocation, does that imply anything about its investment guarantee? The next proposition shows that without further structure, the investment guarantee in our setting can be arbitrarily bad, even if the allocation guarantee is strong.

Proposition 2.4. Let \( \Psi \) be the set of instances such that \(|N| = 2, v \in \mathbb{R}^2_+, \text{ and } A = \emptyset(N)\). If \( \Omega \supseteq \Psi \), then for all \( \beta \in (0, 1) \), there exists an algorithm \( x^\beta \) for \( \Omega \) such that
1. \( x^\beta \) is monotone;
2. \( x^\beta \) is a \( \beta \)-approximation for allocation; and
3. for all \( \beta' > 0 \), \( x^\beta \) is not a \( \beta' \)-approximation for investment.

Note that the setting of Proposition 2.4 includes the knapsack problem, which we define in Section 2.2.3.

Proof of Proposition 2.4. We construct the desired algorithm \( x^\beta \):
\[
x^\beta(v, A) = \begin{cases} 
\{1, 2\} & \text{if } (v, A) \in \Psi \text{ and } \frac{v_1}{v_1 + v_2} < \beta \\
\{1\} & \text{if } (v, A) \in \Psi \text{ and } \frac{v_1}{v_1 + v_2} \geq \beta \\
\text{OPT}(v, A) & \text{otherwise.}
\end{cases}
\]
By inspection, $x^\beta$ is monotone and a $\beta$-approximation for allocation. Moreover, since bidder 1 is always packed for instances in $\Psi$, 1’s threshold price at such instances is 0.

Consider the investment technology $I_1 = \{(\gamma + \epsilon, \gamma), (0, 0)\}$ for $\gamma, \epsilon > 0$. For any $(v, A) \in \Psi$, 1’s best-response at investment instance $(I_1, v_2, A)$ is to choose investment $(\gamma + \epsilon, \gamma)$. For large enough $\gamma$, however, $x^\beta$ packs only bidder 1, for total welfare $\epsilon$. By contrast, the optimal benchmark chooses investment $(\gamma + \epsilon, \gamma)$ and packs both bidders, for total welfare $v_2 + \epsilon$. For all $\beta’ > 0$, we can pick $v_2 > 0$ and small enough $\epsilon$, so $W_{x}(I_1, v_2, A) = \epsilon < \beta’(v_2 + \epsilon) = \beta’W^*(I_1, v_2, A)$. \hfill \Box

### 2.2 Results for binary outcomes

When evaluating each investment technology, we selected the welfare-minimizing best-response (1). Our next result states that an algorithm’s investment approximation ratio over all instances is equal to its approximation ratio over instances with singleton best-responses. Thus, any selection from the best-response correspondence yields the same approximation ratio.

**Lemma 2.1.** If for all $(I_\iota, v_{-\iota}, A)$ such that $\text{BR}(x, I_\iota, v_{-\iota}, A)$ is a singleton, we have

$$\beta W^*(I_\iota, v_{-\iota}, A) \leq W_x(I_\iota, v_{-\iota}, A),$$

then $x$ is a $\beta$-approximation for investment.

**Proof.** We prove the contrapositive: Suppose $x$ is not a $\beta$-approximation for investment. Then there exists some $(I_\iota, v_{-\iota}, A)$ such that

$$\beta W^*(I_\iota, v_{-\iota}, A) > W_x(I_\iota, v_{-\iota}, A).$$

We now modify $I_\iota$ to ensure that $\iota$’s best-response is singleton. Let

$$(\hat{v}_\iota, \hat{c}_\iota) \in \arg\min_{(v_\iota, c_\iota) \in \text{BR}(x, I_\iota, v_{-\iota}, A)} \{W_x(v_\iota, v_{-\iota}, A) - c_\iota\}.$$ 

For $\delta > 0$, let $I_\iota^\delta$ be the investment technology produced by raising by $\delta$ the cost of all investments except $(\hat{v}_\iota, \hat{c}_\iota)$, and then re-normalizing the costs so that

$$\min\{c_\iota : (v_\iota, c_\iota) \in I_\iota^\delta\} = 0.$$ 

Now $\text{BR}(x, I_\iota^\delta, v_{-\iota}, A) = \{(\hat{v}_\iota, \hat{c}_\iota)\}$ by construction, making it a singleton. Moreover, in
constructing $I^δ_i$, each investment’s cost has changed by no more than $δ$. Thus,

$$W^*(I^δ_i, v_{-i}, A) \geq W^*(I_i, v_{-i}, A) - δ$$

$$W_x(I_i, v_{-i}, A) + δ \geq W_x(I^δ_i, v_{-i}, A).$$

For small enough $δ$, we then have

$$βW^*(I^δ_i, v_{-i}, A) > W_x(I^δ_i, v_{-i}, A),$$

which completes the proof of the contrapositive.

We now characterize the investor’s best response facing any threshold auction. In particular, we show that the bidder can find an optimal investment using the following procedure:

1. First, find the investment that would maximize his value net of cost.
2. Make that investment if the associated value net of cost is above the threshold price; otherwise, make a costless investment.

Lemma 2.2. Given an instance $(I_i, v_{-i}, A)$, let $(v^+_i, c^+_i)$ denote an arbitrary element of $\arg\max_{(v_i, c_i) \in I_i} \{v_i - c_i\}$. Let $(v^+_i, c^+_i) \in I_i$ denote a costless investment ($c^+_i = 0$). For any monotone algorithm $x$:

1. if $i \in x(v^+_i - c^+_i, v_{-i}, A)$, then $(v^+_i, c^+_i)$ is a best-response for $i$;
2. otherwise, $(v^+_i, c^+_i)$ is a best-response for $i$.

In Section 3.2, we prove a more general version of Lemma 2.2 (specifically, Lemma 3.1); here, we present an elementary argument for the binary outcome case.

Proof of Lemma 2.2. Let $τ_i(v_{-i}, A)$ be the threshold price for $i$. To reduce clutter, we suppress the dependence of $u_i$, $x_i$, and $τ_i$ on $(v_{-i}, A)$. To prove clause 1, we suppose that $i \in x(v^+_i - c^+_i)$. Then $v^+_i - c^+_i \geq τ_i$, and by $x$ monotone, $i \in x(v^+_i)$. Thus,

$$u_i(v^+_i, c^+_i) = v^+_i - τ_i - c^+_i \geq 0.$$ 

Take any $(v_i, c_i) \in I_i$. We want to prove that $u_i(v^+_i, c^+_i) \geq u_i(v_i, c_i)$. If $u_i(v_i, c_i) \leq 0$, then we are done. If $u_i(v_i, c_i) > 0$, then

$$u_i(v_i, c_i) = v_i - τ_i - c_i \leq v^+_i - τ_i - c^+_i = u_i(v^+_i, c^+_i),$$

where the inequality follows because $(v^+_i, c^+_i) \in \arg\max_{(v_i, c_i) \in I_i} \{v_i - c_i\}$. 

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Now, to prove clause 2, we suppose that $n \notin x(v_i^{\uparrow} - c_i^{\uparrow})$. Take any $(v_i, c_i) \in I_i$. We want to prove that $u_i(v_i^{\uparrow}, c_i^{\uparrow}) \geq u_i(v_i, c_i)$. As $x_i(v_i^{\uparrow} - c_i^{\uparrow}) = 0,$

$$\tau_i \geq v_i^{\uparrow} - c_i^{\uparrow} \geq v_i - c_i.$$

Thus, we have $u_i(v_i, c_i) = \max \{v_i - \tau_i, 0\} - c_i \leq 0 \leq \max \{v_i^{\uparrow} - \tau_i, 0\} = u_i(v_i^{\uparrow}, c_i^{\uparrow}).$ \hfill $\square$

We now introduce a notation for the welfare generated by selecting allocation $a$ at value profile $v$,

$$w(a \mid v) \equiv \sum_{n \in a} v_n.$$

With this notation, note that we have $W_x(v, A) = w(x(v, A) \mid v)$.

We now state the key definition for our main theorem.

**Definition 2.7.** Algorithm $x$ is **ARNIE** (avoiding relevant negative investment externalities) if for any two instances $(v, A)$ and $(\tilde{v}_n, v_{-n}, A)$ of the allocation problem, if

1. either $\tilde{v}_n > v_n$ and $n \in x(v, A),$  
2. or $\tilde{v}_n < v_n$ and $n \notin x(v, A),$ 

then we have

$$w(x(\tilde{v}_n, v_{-n}, A) \mid \tilde{v}_n, v_{-n}) \geq w(x(v, A) \mid \tilde{v}_n, v_{-n})$$

For a monotone algorithm, ARNIE is equivalent to the requirement that if we raise the value of a packed bidder by $\Delta$, then the algorithm’s welfare rises by at least $\Delta$, and if we lower the value of an unpacked bidder, then the algorithm’s welfare does not fall.$^4$

ARNIE algorithms can suffer some kinds of negative investment externalities. For instance, a bidder $n$ could be unpacked at $(v, A)$, but packed at $(\tilde{v}_n, v_{-n}, A)$ for $\tilde{v}_n > v_n$, displacing other bidders so that

$$w(x(\tilde{v}_n, v_{-n}, A) \mid \tilde{v}_n, v_{-n}) < w(x(v, A) \mid \tilde{v}_n, v_{-n})$$

**Theorem 2.1.** Assume $x$ is monotone. If $x$ is ARNIE and is a $\beta$-approximation for allocation, then $x$ is a $\beta$-approximation for investment.

According to Definition 2.7, ARNIE requires that two kinds of investment or disinvestment—“relevant ones”—cause no negative externality on the other bidders. Theorem 2.1 then

---

$^4$The definition of a ‘bitonic’ algorithm in Mu’Alem and Nisan (2008) requires that if we raise the value of a packed bidder, then the algorithm’s welfare does not fall, and if we lower the value of an unpacked bidder, then the algorithm’s welfare does not fall. Every monotone ARNIE algorithm is bitonic, but not vice versa.

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reflects the logic explained in the Introduction, focusing on the ratio of algorithm performance to optimal performance: If \( x \) is ARNIE, then when a bidder increases the value of his \( x \)-allocation by \( \Delta \), the algorithm’s welfare rises by at least \( \Delta \) and the optimal welfare rises by at most \( \Delta \), so the performance ratio can only rise. (A similar argument applies to disinvestment.) The proof of the theorem highlights that only the two investment types in Definition 2.7 can threaten the performance ratio.

### 2.2.1 Proof of Theorem 2.1

By Lemma 2.1, we can restrict attention to instances \((I_\iota, v_\iota, A_\iota)\) with singleton best-responses. To reduce clutter, we suppress the dependence of \( W_{\text{OPT}}, W_x, \) and \( x \) on \( v_\iota \) and \( A_\iota \). Let \((v_\iota^\uparrow, c_\iota^\uparrow)\) denote an arbitrary element of \( \arg\max_{(v_\iota, c_\iota) \in I_\iota} \{v_\iota - c_\iota\} \), and let \((v_\iota^\downarrow, c_\iota^\downarrow)\) denote a costless investment \((c_\iota^\downarrow = 0)\).

By Lemma 2.2, there are two cases to consider. Either \( \iota \) chooses \((v_\iota^\uparrow, c_\iota^\uparrow)\) and \( \iota \in x(v_\iota^\uparrow - c_\iota^\uparrow) \), or \( \iota \) chooses \((v_\iota^\downarrow, c_\iota^\downarrow)\) and \( \iota \notin x(v_\iota^\uparrow - c_\iota^\uparrow) \). The next two inequalities below follow from the hypothesis that \( x \) is ARNIE.

If \( \iota \) chooses \((v_\iota^\uparrow, c_\iota^\uparrow)\) and \( \iota \in x(v_\iota^\uparrow - c_\iota^\uparrow) \), then as \( x \) is ARNIE,

\[
W_x(I_\iota) = W_x(v_\iota^\uparrow) - c_\iota^\uparrow \geq W_x(v_\iota^\uparrow - c_\iota^\uparrow).
\]

If \( \iota \) chooses \((v_\iota^\uparrow, c_\iota^\uparrow)\) and \( \iota \notin x(v_\iota^\uparrow - c_\iota^\uparrow) \), then as \( x \) is ARNIE,

\[
W_x(I_\iota) = W_x(v_\iota^\downarrow) - c_\iota^\downarrow = W_x(v_\iota^\uparrow - c_\iota^\downarrow) \geq W_x(v_\iota^\uparrow - c_\iota^\downarrow).
\]

Let \((v_\iota^*, c_\iota^*)\) be an element of \( \arg\max_{(v_\iota, c_\iota) \in I_\iota} \{W^*(v_\iota) - c_\iota\} \), so that

\[
\overline{W^*}(I_\iota) = W^*(v_\iota^*) - c_\iota^* = W^*(v_\iota^* - c_\iota^*) \leq W^*(v_\iota^\uparrow - c_\iota^\downarrow).
\]

Thus, as \( x \) is a \( \beta \)-approximation for allocation, we have

\[
\overline{W}_x(I_\iota) \geq W_x(v_\iota^\uparrow - c_\iota^\downarrow) \geq \beta W^*(v_\iota^\uparrow - c_\iota^\downarrow) \geq \beta \overline{W^*}(I_\iota).
\]

This completes the proof of Theorem 2.1.

### 2.2.2 Non-bossiness and ARNIE

**Definition 2.8.** Algorithm \( x \) is **non-bossy** if for all \((v, A)\) and \( \tilde{v}_n \), if \( x_n(v, A) = x_n(\tilde{v}_n, v_{-n}, A) \), then \( x(v, A) = x(\tilde{v}_n, v_{-n}, A) \), that is, if no bidder can affect other bidders’ outcomes without affecting his own.
**Proposition 2.5.** If $x$ is monotone and non-bossy, then $x$ is ARNIE.

**Proof.** Take any two instances $(v, A)$ and $(\tilde{v}_n, v_n, A)$ that satisfy the antecedent condition of Definition 2.7. As $x$ is monotone, we have $x_n(v, A) = x_n(\tilde{v}_n, v_n, A)$. Then, as $x$ is non-bossy, we have $x(v, A) = x(\tilde{v}_n, v_n, A)$. Thus, we see that

$$w(x(v, A) | \tilde{v}_n, v_n) = w(x(\tilde{v}_n, v_n, A) | \tilde{v}_n, v_n),$$

as desired. \qed

ARNIE requires that for some of the value changes for an individual that do not affect that individual’s outcome, $x$ should not pick less valuable outcomes for others. Non-bossiness is stronger: it requires that any value changes for an individual that do not affect that individual’s outcome, $x$ should not make any change in others’ outcomes.

**Proposition 2.6.** Let $X$ be a collection of ARNIE algorithms. If $y$ is an algorithm that at each instance $(v, A) \in \Omega$ outputs a surplus-maximizing allocation from the collection \{ $x(v, A)$ \}_{x \in X}, then $y$ is ARNIE.

**Proof.** We consider any two instances $(v, A)$ and $(\tilde{v}_n, v_n, A)$ satisfying the antecedent condition of Definition 2.7. Let $x \in X$ be such that $y(v, A) = x(v, A)$. As $x$ is ARNIE, we have

$$w(y(v, A) | \tilde{v}_n, v_n) = w(x(v, A) | \tilde{v}_n, v_n) \leq w(x(\tilde{v}_n, v_n, A) | \tilde{v}_n, v_n) \leq w(y(\tilde{v}_n, v_n, A) | \tilde{v}_n, v_n),$$

as desired. \qed

**2.2.3 Application: Knapsack algorithms**

The knapsack problem is a special case of the allocation problem introduced in Section 2.1.1. In the knapsack problem, there is a set of items, where an item $n$ has value $v_n$ and size $s_n$. The knapsack has capacity $S$. Without loss of generality, suppose no item’s size is more than $S$. The set of feasible allocations is any subset of items $K \subseteq N$ such that $\sum_{n \in K} s_n \leq S$. As before, let $A$ denote the set of feasible allocations and let $a$ be an element of $A$.

The knapsack problem is NP-Hard (Karp, 1972); there is no known polynomial-time algorithm that outputs optimal allocations (Cook, 2006; Fortnow, 2009). Dantzig (1957) suggested applying a **Greedy algorithm** to the knapsack problem. Formally:
**Algorithm 1** (Greedy). Sort items by the ratio of their values to their sizes so that

\[ \frac{v_1}{s_1} \geq \frac{v_2}{s_2} \geq \cdots \geq \frac{v_{|N|}}{s_{|N|}} \]  

(3)

Add items to the knapsack one by one in the sorted order so long as the sum of the sizes does not exceed the knapsack’s capacity. When encountering the first item that would violate the size constraint, stop.

Although Dantzig’s Greedy algorithm performs well on some instances, including ones for which all items are small in relation to the size of the knapsack, its worst-case performance guarantee is 0, as illustrated by the following example.

**Example 2.1.** Consider a Knapsack with capacity 1 and two items. For some arbitrarily small positive \( \epsilon \), let \( v_1 = \epsilon, s_1 = \frac{\epsilon}{2}, v_2 = 1, \) and \( s_2 = 1 \). The Greedy algorithm picks item 1 and stops, whereas the optimal algorithm picks item 2. Thus, Greedy’s performance is no better than \( \epsilon \) of the optimum.

There is a simple modification of the Greedy algorithm that improves the worst-case guarantee for the knapsack problem. Let us define the **Modified Greedy** algorithm as follows.

**Algorithm 2** (Modified Greedy). *Run the Greedy algorithm. Compare the Greedy Algorithm’s packing to the most valuable individual item; output whichever has higher welfare.*

Modified Greedy’s worst-case performance is much better than Greedy’s:

**Proposition 2.7.** Modified Greedy is a \( \frac{1}{2} \)-approximation for the Knapsack problem.

*Proof.* For any instance \( \omega \), order the items by value/size as in (3). Let \( k \) be the lowest index of an item not packed by Greedy and let \( K \) be the index of the most valuable item that Greedy fails to pack. We have

\[
W_{\text{OPT}}(\omega) \leq \sum_{j=1}^{k} v_j = W_{\text{Greedy}}(\omega) + v_k \\
\leq W_{\text{Greedy}}(\omega) + v_K \\
\leq 2 \max \{W_{\text{Greedy}}(\omega), v_K\} \\
= 2W_{\text{ModifiedGreedy}}(\omega).
\]

For the knapsack problem, there is in fact a **fully polynomial time approximation scheme** (FPTAS) that, for any \( \epsilon > 0 \), yields a \( (1 - \epsilon) \)-approximation, and runs in polynomial time in
both the number of items and $\frac{1}{\epsilon}$. The standard FPTAS algorithm rounds down the values, and uses dynamic programming to output an optimal allocation for the rounded instance. For details, we refer interested readers to Williamson and Shmoys (2011, p. 65-68) or Vazirani (2013, p. 68-70). 

All of the algorithms we have just described are ARNIE.

**Proposition 2.8.** For the knapsack problem, the Greedy algorithm, the Modified Greedy algorithm, and the standard FPTAS all are ARNIE.

**Proof.** The Greedy algorithm is a monotone and non-bossy algorithm, and thus it is ARNIE by Proposition 2.5.

The Modified Greedy algorithm’s output is equal to the welfare-maximizing selection from the outputs of two algorithms:

- the Greedy algorithm, and

- the algorithm that selects the most valuable single item.

We have just shown that the Greedy algorithm is ARNIE. Meanwhile, the algorithm that selects the most valuable single item is monotone and non-bossy and so is ARNIE by Proposition 2.5, as well. Thus, by Proposition 2.6, the Modified Greedy algorithm is ARNIE.

The standard FPTAS is non-bossy on the rounded instance. Moreover, changing one bidder’s value does not affect the algorithm’s output unless it changes the rounded instance. Therefore, the standard FPTAS is non-bossy, and by Proposition 2.5 it is ARNIE.

For the example in the Introduction, the Greedy and Modified Greedy algorithms output the same packings. Hence, that example shows that there can be negative investment externalities under the Modified Greedy algorithm. In particular, an investment that causes the investor to be packed can increase the investor’s utility, but yield a reduction in social welfare. However, these externalities are not relevant, in the sense that they do not undermine the Modified Greedy algorithm’s worst-case performance guarantee of $\frac{1}{2}$.

**2.2.4 A weaker sufficient condition**

Definition 2.7 is sufficient for approximation guarantees to persist under investment; it is not necessary, however—the following weaker condition will do.

**Definition 2.9.** Algorithm $x$ is weakly ARNIE if for any two instances $(v, A)$ and $(\tilde{v}_n, v_{-n}, A)$ of the allocation problem, if

1. either $\tilde{v}_n > v_n$ and $n \in x(v, A)$,
2. or $\tilde{v}_n < v_n$ and $n \notin x(v, A)$ and $n \in a$ for all $a \in \arg\max_{a' \in A} w(a' | v)$, then we have

$$w(x(\tilde{v}_n, v - n, A) | \tilde{v}_n, v - n) \geq w(x(v, A) | \tilde{v}_n, v - n).$$

Intuitively, $x$ is weakly ARNIE if two conditions hold. First, if $n$ is selected by $x$ at $v$, then increasing his value increases the welfare achieved by $x$ by at least an equal amount. This is the same as the first ARNIE condition. Second, if $n$ is not selected by $x$ at $v$ but is part of every optimal solution at $v$, then decreasing his value does not reduce the welfare achieved by $x$. This weakens the second ARNIE condition, requiring it to hold only if the argmax condition is satisfied.

Clause 2 of Definition 2.9 equivalently requires that $\tilde{v}_n \leq v_n$ if $n \notin x(v, A)$ and for all $\epsilon > 0$ we have $W^*(v_n - \epsilon, v - n, A) < W^*(v_n, v - n, A)$.

**Theorem 2.2.** Assume $x$ is monotone. If $x$ is weakly ARNIE and is a $\beta$-approximation for allocation, then $x$ is a $\beta$-approximation for investment.

Theorem 2.2 establishes that ARNIE is not a necessary condition for worst-case guarantees to persist under investment, as weak ARNIE is sufficient. However, in problems of interest, there is no known fast method to compute optimal allocations. Thus, Clause 2 of Definition 2.9 may be intractable to verify.

**Definition 2.10.** For two problems $\Omega$ and $\Omega'$, $\Omega'$ is a sub-problem of $\Omega$ if $\Omega' \subseteq \Omega$.

If $x$ is monotone and weakly ARNIE on $\Omega$, then $x$ is monotone and weakly ARNIE on any sub-problem $\Omega'$; thus, we obtain the following corollary of Theorem 2.2.

**Corollary 2.2.** Suppose that $x$ is monotone and is weakly ARNIE on problem $\Omega$. For any sub-problem $\Omega'$, if $x$ is a $\beta'$-approximation for allocation on $\Omega'$, then $x$ is a $\beta'$-approximation for investment on $\Omega'$.

### 2.2.5 A maximal-domain theorem

We now show that weak ARNIE comprises a maximal domain for allocative guarantees to extend to investment guarantees.

**Theorem 2.3.** Assume $x$ is monotone and a $\beta$-approximation for allocation on problem $\Omega$ for $\beta > 0$. Suppose that for all $(v - \epsilon, A)$, there exists a partition of $V^A_\epsilon$ into positive-length intervals such that $x(\cdot, v - \epsilon, A)$ is measurable with respect to that partition.

If $x$ is not weakly ARNIE, then there exists a sub-problem $\Omega' \subseteq \Omega$ and $\beta'$ such that $x$ is a $\beta'$-approximation for allocation on $\Omega'$, but not a $\beta'$-approximation for investment on $\Omega'$. 

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2.3 Allowing multiple investors

We now show how to extend our framework and approach to a setting in which multiple bidders are able to invest.

An instance of the multi-investor problem is a tuple \((I,A)\), where \(I = (I_n)_{n \in \mathbb{N}}\) and \(I_n \subseteq V_n^A \times \mathbb{R}\) is a set of feasible investments. We restrict attention to investment technologies that satisfy:

1. **Finite.** \(|I_n| < \infty\).

2. **Normalization.** \(\min \{c_n : (v_n, c_n) \in I_n\} = 0\).

When multiple investors simultaneously choose investments, even Vickrey auctions can suffer from inefficient investments due to a coordination problem, as the following example illustrates.

**Example 2.2.** Consider the knapsack problem. There is a knapsack with capacity 2, and three bidders, with sizes \(s_1 = 2\), \(s_2 = s_3 = 1\). Bidder 1 has the singleton technology \(I_1 = \{(10,0)\}\). Bidders 2 and 3 have the technology \(I_2 = I_3 = \{(0,0), (9,1)\}\). It is socially optimal for Bidders 2 and 3 to both choose \((9,1)\) and both be packed. However, if only one of the bidders invests, it will not be packed. In the Vickrey auction \((\text{OPT}, p^\text{OPT})\), there are two Nash equilibrium investment profiles. In one Nash equilibrium, no bidder invests. In the efficient Nash equilibrium, both bidders 2 and 3 invest.

It is unclear whether ARNIE is enough, in general, to ensure that an efficient Nash equilibrium exists. However, if the algorithm is monotone and non-bossy and guarantees a fraction \(\beta\) in the short-run problem, then even with multiple investors, there is an equilibrium of the long-run problem that achieves the same performance.

**Theorem 2.4.** Assume that \(x\) is monotone, non-bossy, and a \(\beta\)-approximation for allocation. For any instance of the multi-investor problem \((I,A)\), there exists a Nash equilibrium \((\hat{v}, \hat{c})\) of the investment game facing threshold auction \((x, p^x)\), such that:

\[
W_x(\hat{v}, A) - \sum_n \hat{c}_n \geq \beta \max_{(v,c) \in I} \left\{ W^*(v, A) - \sum_n c_n \right\}.
\]

3 Investment with multiple outcomes

The problems we studied in Section 2 were generalizations of the knapsack problem: each bidder is either packed or unpacked. We now generalize Theorem 2.1 to a setting in which each bidder can have more than two potential outcomes.
3.1 Allocation problems with multiple outcomes

Now, \( O \) denotes a finite set of outcomes. Each bidder’s value \( v_n \in \mathbb{R}^O \) is a row vector, with element \( v^o_n \) denoting \( n \)'s value for outcome \( o \). We normalize the value of one outcome \( o \), \( v^o_n = 0 \); this is \( n \)’s value for “being unpacked.” A value profile \( v = (v_n)_{n \in N} \) specifies a value for each bidder.

An allocation \( a = (a_n)_{n \in N} \) specifies an outcome \( a_n \in O \) for each bidder \( n \). It is convenient to represent \( a_n \) as a binary vector, with \( a^o_n = 1 \) if \( o \) is the outcome for bidder \( n \), and 0 otherwise.

An instance \((v,A)\) consists of a value profile \( v \) and a set of \( A \) of feasible allocations, such that for all \( a \in A \), \( v \)’s dimensions agree with \( a \)’s dimensions. We assume that the allocation in which every bidder is unpacked is feasible.

An allocation problem consists of a collection of instances, denoted \( \Omega \). For each \( A \) and \( n \), let \( V^A_n \subseteq \mathbb{R}^O \) denote the space of possible value vectors for bidder \( n \). We assume a product structure: for all \( A \), \( \{v : (v,A) \in \Omega \} = \prod_n V^A_n \).

The welfare generated by selecting allocation \( a \in A \) at instance \((v,A)\) is

\[
w(a \mid v) \equiv \sum_n a_n \cdot v_n.
\]

As before, an algorithm \( x \) selects, for each instance \((v,A) \in \Omega \), a feasible allocation \( x(v,A) \in A \); we denote \( n \)'s outcome under \( x \) at \((v,A)\) by \( x_n(v,A) \). Welfare of algorithm \( x \) at instance \((v,A)\) is

\[
W_x(v,A) \equiv w(x(v,A) \mid v).
\]

3.2 Reporting problems with multiple outcomes

A mechanism \((x,p)\) consists of an algorithm \( x \) with \( x(v,A) \in A \) and a payment rule \( p \) with \( p(v,A) \in \mathbb{R}^N \). With multiple outcomes, it is less straightforward to characterize the strategy-proof mechanisms. A necessary condition is weak monotonicity of \( x \).

Definition 3.1. \( x \) is weakly monotone (W-Mon) if for any two instances \((v_n,v_{-n},A)\) and \((\bar{v}_n,v_{-n},A)\), we have

\[
\bar{v}_n \cdot x_n(\bar{v}_n,v_{-n},A) - \bar{v}_n \cdot x_n(v_n,v_{-n},A) \geq v_n \cdot x_n(\bar{v}_n,v_{-n},A) - v_n \cdot x_n(v_n,v_{-n},A).
\]

Proposition 3.1 (Lavi et al. (2003)). If there exists \( p \) such that \((x,p)\) is strategy-proof, then \( x \) is W-Mon.
Moreover, as in our setting each $V^A_n$ is convex, W-Mon is also a sufficient condition.\footnote{Bikhchandani et al. (2006) provide other domain assumptions such that W-Mon is sufficient.}

**Proposition 3.2** (Saks and Yu (2005)). *If for all $n$ and $A$, the set of possible values $V^A_n$ is convex, then if $x$ is W-Mon, there exists $p$ such that $(x,p)$ is strategy-proof.*

### 3.3 Investment problems with multiple outcomes

As before, we suppose that a bidder $\iota \in N$ has the opportunity to invest before reporting and allocation. An **investment** is a pair $(v_\iota, c_\iota)$, with $v_\iota \in (\mathbb{R}^+_0)^O$ and $c_\iota \in \mathbb{R}$. An **investment instance** is a tuple $(I_\iota, v_{-\iota}, A)$, where $I_\iota \subseteq V^A_\iota \times \mathbb{R}$ is a set of feasible investments and $v_{-\iota} \in V^A_\iota$. We restrict attention to investment instances that satisfy:

1. **Finite.** $|I_\iota| < \infty$.
2. **Normalization.** $\min \{c_\iota : (v_\iota, c_\iota) \in I_\iota\} = 0$.

Given any W-Mon algorithm $x$, we suppose that $\iota$ faces a strategy-proof mechanism $(x, p^x)$. We define $u_\iota(\cdot), BR(\cdot), \overline{W}_x(\cdot)$, and $\overline{W}^*(\cdot)$ as before. Note that for convex $V^A_\iota$, the particular choice of payment rule does not matter—$V^A_\iota$ is path-connected, so by the envelope theorem, $\iota$’s best-responses are the same for all strategy-proof payment rules (Milgrom and Segal, 2002).

### 3.4 Results for multiple outcomes

We now generalize our ARNIE condition to allow for more than two outcomes.

**Definition 3.2.** Algorithm $x$ is **ARNIE** if for any two instances $(v, A)$ and $(\tilde{v}_n, v_{-n}, A)$, if for all outcomes $o$:

$$
\tilde{v}^{x_n(v,A)} - \tilde{v}^o_n \geq v^{x_n(v,A)} - v^o_n,
$$

then

$$
W(x(\tilde{v}_n, v_{-n}, A) \mid \tilde{v}_n, v_{-n}) \geq W(x(v, A) \mid \tilde{v}_n, v_{-n}).
$$

**Theorem 3.1.** Assume that $x$ is W-Mon and that $V^A_n$ is a product of one-dimensional intervals for all $A$ and $n$. If $x$ is ARNIE and is a $\beta$-approximation for allocation, then $x$ is a $\beta$-approximation for investment.

Theorem 3.1 extends our earlier result to a much more general model that includes multiple outcomes. Here, what we have called the “relevant” investments for a bidder are those that increase the bidder’s marginal value for his original outcome compared to every
other outcome. Theorem 3.1 tells us that if there are no negative externalities associated with relevant investments, then the long-run problem inherits the worst-case guarantee from the short-run problem.

3.4.1 Proof of Theorem 3.1

As in the theorem statement, we suppose that \( x \) is W-Mon, ARNIE, and a \( \beta \)-approximation for allocation and that each \( V^A_n \) is a product of one-dimensional intervals. We define a pivotal vector \( \pi_i \) that plays a key role in our argument. For each \( o \in O \), the corresponding component of the pivotal vector is

\[
\pi_i^o = \max_{(v_i,c_i) \in I_i} \{ v_i^o - c_i \}. \tag{4}
\]

As \( I_i \) is normalized and \( V^A_i \) is a product of one-dimensional intervals, we have \( \pi_i \in V^A_i \) by construction.

We begin by showing that the investor \( i \) can find a best-response using the following simple procedure:

1. Construct the pivotal vector \( \pi_i \)

2. Check what outcome would occur if he reported the pivotal vector to the mechanism, this is \( x_i(\pi_i, v_{-i}, A) \).

3. Choose an investment that maximizes his value, net of costs, for \( x_i(\pi_i, v_{-i}, A) \).

The next lemma formalizes this procedure.

Lemma 3.1. For any instance \((I_i, v_{-i}, A)\), it is a best-response for \( i \) to choose \((v_i, c_i)\) to maximize

\[
\max_{(v_i,c_i) \in I_i} \{ v_i \cdot x_i(v_i) - p_i^x(v_i) - c_i \}. \tag{5}
\]

Proof. Bidder \( i \)'s best response corresponds to the maximization

\[
\max_{(v_i,c_i) \in I_i} \{ v_i \cdot x_i(v_i) - p_i^x(v_i) - c_i \}. \tag{5}
\]

As \((x, p^x)\) is strategy-proof,

\[
v_i \cdot x_i(\bar{v}_i) - p_i^x(\bar{v}_i)
\]

is maximized by taking \( \bar{v}_i = v_i \); hence, we can rewrite the maximand in (5) to yield

\[
\max_{(v_i,c_i) \in I_i} \max_{v_i} \{ v_i \cdot x_i(\bar{v}_i) - p_i^x(\bar{v}_i) - c_i \}. \tag{6}
\]
Changing the order of maximization in (6) then gives us

$$\max_{\tilde{v}_i} \max_{(v_i, c_i) \in I_i} \{v_i \cdot x_i(\tilde{v}_i) - p^*_i(\tilde{v}_i) - c_i\}.$$  

Now, by our construction of $\overline{v}_i$, for all $\tilde{v}_i \in V^A_i$, we have

$$\max_{(v_i, c_i) \in I_i} \{v_i \cdot x_i(\tilde{v}_i) - p^*_i(\tilde{v}_i) - c_i\} = \overline{v}_i \cdot x_i(\tilde{v}_i) - \overline{p}_i(\tilde{v}_i), \tag{7}$$

as $x_i(\tilde{v}_i) \in O$. As $(x, p^*)$ is strategy-proof, setting $\tilde{v}_i = \overline{v}_i$ maximizes the right-hand side of (7). This reduces $i$’s problem to the maximization

$$\max_{(v_i, c_i) \in I_i} \{v_i \cdot x_i(\overline{v}_i) - \overline{p}_i(\overline{v}_i) - c_i\} = \max_{(v_i, c_i) \in I_i} \{v_i \cdot x_i(\overline{v}_i) - \overline{c}_i\}. \tag{8}$$

Dropping the term in (8) that does not depend on $(v_i, c_i)$ yields

$$\max_{(v_i, c_i) \in I_i} \{v_i \cdot x_i(\overline{v}_i) - c_i\},$$

which gives us Lemma 3.1.

**Lemma 3.2.** For any instance $(I_i, v_{-i}, A)$, we have

$$\overline{W}^*(I_i, v_{-i}, A) = W^*(\overline{v}_i, v_{-i}, A).$$

**Proof.**

$$\overline{W}^*(I_i, v_{-i}, A) = \max_{(v_i, c_i) \in I_i} \max_{a \in A} \{w(a \mid v_i, v_{-i}) - c_i\}$$

$$= \max_{a \in A} \max_{(v_i, c_i) \in I_i} \{w(a \mid v_i, v_{-i}) - c_i\}$$

$$= \max_{a \in A} \{w(a \mid \overline{v}_i, v_{-i})\}$$

$$= W^*(\overline{v}_i, v_{-i}, A). \qed$$

Now, with Lemma 3.1 and Lemma 3.2, we can proceed with the proof of Theorem 3.1. By the same argument as in the proof of Lemma 2.1, we can restrict attention to proving the desired bound for instances with singleton best-responses. We let $(\hat{v}_i, \hat{c}_i) \in BR(x_i, I_i, v_{-i}, A)$ denote $i$’s best-response.

We now prove that moving from $\overline{v}_i$ to $\hat{v}_i$ satisfies the antecedent condition of Defini-
tion 3.2: For all outcomes \( o \), we have

\[
\hat{v}^x_i(\vec{v}_i) - \hat{v}_i^o = (\hat{v}^x_i(\vec{v}_i) - \hat{c}_i) - (\hat{v}_{i}^o - \hat{c}_i) \\
\geq \max_{(v_i, c_i) \in I_i} \{ v_i^x(\vec{v}_i) - c_i \} - \max_{(v_i, c_i) \in I_i} \{ v_i^o - c_i \} \\
= \pi^x_i(\vec{v}_i) - \pi_i^o,
\]

where the inequality follows from Lemma 3.1, given that \((\hat{v}_i, \hat{c}_i) \in \text{BR}(x, I_i, v_{-i}, A)\) is a best response. Thus, as \( x \) is ARNIE, we have that

\[
W_{x}(\hat{v}_i) = w(x(\hat{v}_i) \mid \hat{v}_i) \geq w(x(\vec{v}_i) \mid \hat{v}_i).
\] (9)

Now, by our construction of the pivotal vector \( \vec{v}_i \) in (4) and by Lemma 3.1, we have

\[
\hat{v}^x_i(\vec{v}_i) - \hat{c}_i = \pi^x_i(\vec{v}_i)
\]

which implies

\[
w(x(\vec{v}_i) \mid \hat{v}_i) - \hat{c}_i = w(x(\vec{v}_i) \mid \vec{v}_i) = W_{x}(\vec{v}_i).
\] (10)

Subtracting \( \hat{c}_i \) from (9) and applying (10), we find that

\[
W_{x}(\hat{v}_i) - \hat{c}_i \geq W_{x}(\vec{v}_i).
\] (11)

Combining the preceding steps, we see that

\[
\underbrace{W_{x}(I_i)}_{\text{Lemma 3.2}} = \underbrace{W_{x}(\hat{v}_i) - \hat{c}_i}_{\beta\text{-approx for allocation}} \geq \underbrace{W_{x}(\vec{v}_i)}_{\beta W^*(\vec{v}_i)} = \beta W^*(I_i),
\]

which shows that \( x \) is an approximation for investment, as desired.

### 3.5 Combinatorial auctions

Theorem 3.1 relies on each bidder’s values for different outcomes having a product structure. In a combinatorial auction, an outcome consists of a bundle of goods—and many standard classes of value functions do not have a product structure on bundles. For instance, if a bidder’s value function is additive, then knowing his value for each singleton bundle exactly pins down his value for the grand bundle. Consequently, Theorem 3.1 has limited applicability for combinatorial auctions. Nevertheless, we are able to develop an extension that accommodates a standard class of preferences for combinatorial auctions.
An allocation instance consists of:

1. a finite set of bidders \( N \);
2. a finite set of goods \( G \); and
3. for each \( n \in N \), a value function \( v_n : \wp(G) \to \mathbb{R} \).

We write \( v \) for a profile of value functions; \((v,G)\) denotes an instance. An allocation problem \( \Omega \) is a collection of allocation instances. An algorithm \( x \) selects for each \((v,G)\) a bundle of goods, one for each bidder, \( x(v,G) \in (\wp(G))^N \). We require that no good is allocated twice, that is, for all \( n \neq n' \), we have \( x_n(v,G) \cap x_{n'}(v,G) = \emptyset \).

Correspondingly, an investment instance consists of:

1. a cost function for the investing bidder, \( c_\iota : V_\iota \to \mathbb{R} \), for some domain of value functions \( V_\iota \);
2. a profile of value functions for the other bidders, \( v_{-\iota} \); and
3. a set of goods \( G \).

As before, the investing bidder \( \iota \) faces a strategy-proof mechanism \((x,p^x)\), and chooses an investment \( v_\iota \in V_\iota \).

When value functions are fully general, a bidder’s preferences are described by \(|\wp(G)|\) real numbers, and it is computationally infeasible even to approximate the optimum. Hence, we study allocation and investment under fractionally subadditive value functions. These are a canonical class of preferences, for which there are known allocation algorithms with non-trivial guarantees (Nisan, 2000; Feige, 2009). The class includes all submodular functions, as well as all functions that have the gross substitutability property (Lehmann et al., 2006; Paes Leme, 2017).

**Definition 3.3.** Value function \( v_n(\cdot) \) is additive if there exists \( \alpha \in (\mathbb{R}_0^+)^G \) such that for all \( F \subseteq G \),

\[
v_n(F) = \sum_{g \in F} \alpha_g.
\]

In the case that a bidder’s value function is additive with parameter vector \( \alpha \), we abuse notation, and use \( \alpha \) to denote the value function itself.

Value function \( v_n(\cdot) \) is fractionally sub-additive (XOS) if there exists a family of additive value functions \((\alpha^\ell)_{\ell \in L}\) such that for all \( F \subseteq G \),

\[
v_n(F) = \max_{\ell} \alpha^\ell(F).
\]

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We denote by $XOS$ the set of all XOS value functions.

We restrict attention to allocation problems such that bidders can have any XOS preferences, that is, for all $(v_n, G)$,

$$\{v_n : (v_n, v_{-n}, G) \in \Omega\} = XOS.$$ 

We restrict attention to cost functions $c_i$ such that, for each investment instance $(c_i, v_{-n}, G)$:

1. The investor’s best-response set is non-empty.
2. The set of socially optimal investments is non-empty.
3. $V_i = XOS$.
4. If for all $F \subseteq G$, $v_i(F) = 0$, then $c_i(v_i) = 0$.

**Definition 3.4.** Cost function $c_i(\cdot)$ is isotone if for any $v_i, \tilde{v}_i \in V_i$, if $v_i(F) \geq \tilde{v}_i(F)$ for all $F \subseteq G$, then $c_i(v_i) \geq c_i(\tilde{v}_i)$.

**Definition 3.5.** For any $\alpha, \alpha' \in (\mathbb{R}^+_0)^G$, let $\alpha \lor \alpha' = (\max\{\alpha_g, \alpha'_g\})_{g \in G}$, and let $\alpha \land \alpha' = (\min\{\alpha_g, \alpha'_g\})_{g \in G}$. Cost function $c_i(\cdot)$ is supermodular on additive valuations if for any $\alpha, \alpha' \in (\mathbb{R}^+_0)^G$ we have

$$c_i(\alpha \lor \alpha') + c_i(\alpha \land \alpha') \geq c_i(\alpha) + c_i(\alpha').$$

We extend the definitions of W-Mon and ARNIE to combinatorial auctions, by regarding each bundle of goods as an outcome.

**Theorem 3.2.** Assume that $x$ is W-Mon, and restrict $c_i$ to be isotone and supermodular on additive valuations. If $x$ is ARNIE and is a $\beta$-approximation for allocation, then $x$ is a $\beta$-approximation for investment.

**Proof.** Given some investment instance $(c_i, v_{-i}, G)$, let the pivotal value function $\tau_i$ be defined by

$$\tau_i(F) \equiv \max_{v_i \in XOS} \{v_i(F) - c_i(v_i)\}$$

for all $F \subseteq G$.

We first derive an analog of Lemma 3.1 for the combinatorial auction setting.

**Lemma 3.3.** If $c_i(\cdot)$ is isotone and supermodular on additive valuations, then $\tau_i \in XOS$.
We once again suppress the dependence of functions on \( v_{-i} \) and \( G \).

We now note that, by the same argument as in Lemma 3.1, in any instance \((c_i, v_{-i}, G)\), choosing \( \hat{v}_i \) to maximize \( v_i(x_i(\bar{v}_i)) - c_i(v_i) \) is a best-response for \( i \). And by the same argument as in Lemma 2.1, we can restrict attention to proving the bound for instances with singleton best-response sets.

By Lemma 3.3, \( \bar{v}_i \in XOS \). Thus, as \( x \) is a \( \beta \)-approximation for allocation, \( W_x(\bar{v}_i) \geq \beta W^*(\bar{v}_i) \). Moreover, just as in the proof of Theorem 3.1, the fact that \( x \) is ARNIE implies that

\[
W_x(\hat{v}_i) - \hat{c}_i \geq W_x(\bar{v}_i).
\]

We then have

\[
W_x(c_i) = W_x(\hat{v}_i) - c_i(\hat{v}_i) \geq W_x(\bar{v}_i) \geq \beta W^*(\bar{v}_i) = \beta W^*(c_i),
\]

which completes the proof.

4 Discussion

Here are some open questions raised by our work:

- Our analysis so far has focused on investment under full information. How, if at all, does the analysis extend to incomplete information? What properties must an allocation algorithm have to retain its performance when a bidder invests without knowing the values of the other bidders? Can the relevant information be elicited in advance through an appropriate choice of mechanism?
- We have analyzed deterministic algorithms. Does the analysis extend to randomized algorithms, with an appropriate generalization of ARNIE?
- How hard is it to generate good investment incentives? Does requiring an allocation algorithm to be ARNIE raise new computational hurdles? In particular, given oracle access to some monotone allocation algorithm, is there a polynomial-time procedure that outputs a monotone ARNIE allocation algorithm with a weakly better approximation ratio?


A Proofs omitted from the main text

Proof of Theorem 2.2

Proof. The proof of Theorem 2.1 established that

$$\overline{W}(I, v_{-i}, A) \geq \beta \overline{W}^*(I, v_{-i}, A)$$

(13)

in two cases:

1. \(i\) chooses \((v_i^+, c_i^+)\) and \(i \in x(v_i^+ - c_i^+)\); and

2. \(i\) chooses \((v_i^+, c_i^+)\) and \(i \notin x(v_i^+ - c_i^+)\).

To establish (13) under the assumption that \(x\) is weakly ARNIE, we consider three cases:

1. \(i\) chooses \((v_i^+, c_i^+)\) and \(i \in x(v_i^+ - c_i^+)\);

2a. \(i\) chooses \((v_i^+, c_i^+)\), \(i \notin x(v_i^+ - c_i^+)\), and \(i \in a\) for all \(a \in \arg\max_{a \in A} \{ w(a \mid v_i^+ - c_i^+, v_{-i}) \} \);
2b. \( \iota \) chooses \((v^1_i, c^1_i)\), \( \iota \not\in x(v^1_i - c^1_i) \), and there exists \( a \in \arg\max_{a \in A} \{w(a \mid v^1_i - c^1_i, v_{-i})\} \) such that \( \iota \not\in a \).

When \( x \) is weakly ARNIE, the same arguments as in the proof of Theorem 2.1 work for Case 1 and Case 2a. Meanwhile, we observe that in Case 2b:

\[
\bar{W}_x(I_v, v_{-i}, A) = W_x(v^1_i, v_{-i}, A) \geq \beta W^*(v^1_i, v_{-i}, A) \geq \beta W^*(v^1_i - c^1_i, v_{-i}, A) \geq \beta \bar{W}^*(I_v, v_{-i}, A),
\]

where the last inequality follows by (2).

\[\Box\]

**Proof of Theorem 2.3**

**Definition A.1.** \( W_x(\cdot, v_{-i}, A) \) is lower semi-continuous at \( v_i \) if for all sequences \( \{v^k_i\}_{k=1}^\infty \) such that \( v^k_i \to v_i \), we have

\[
\limsup_{v^k_i \to v_i} \{W_x(v^k_i, v_{-i}, A)\} \geq W_x(v_i, v_{-i}, A).
\]

**Lemma A.1.** Assume \( x \) is monotone and a \( \beta \)-approximation for allocation on problem \( \Omega \) for \( \beta > 0 \). Assume \( W_x(\cdot, v_{-i}, A) \) is lower semi-continuous at \( v_i \). If there exists \( \tilde{v}_i \) such that \((v_i, A)\) and \((\tilde{v}_i, v_{-i}, A)\) do not satisfy the requirements of Definition 2.9, then there exists a sub-problem \( \Omega' \subseteq \Omega \) and \( \beta' \) such that \( x \) is a \( \beta' \)-approximation for allocation on \( \Omega' \), but not a \( \beta' \)-approximation for investment on \( \Omega' \).

**Proof.** Suppose we have some \((v_i, A)\) and \( \tilde{v}_i \) that do not satisfy the requirements of Definition 2.9. As usual, we will suppress the dependence of functions on \( v_{-i} \) and \( A \). Let

\[
\Omega' = \{ (v'_i, v_{-i}, A) : v'_i \in [\min\{v_i, \tilde{v}_i\}, \max\{v_i, \tilde{v}_i\}] \}
\]

\[\bar{\beta} = \sup\{\beta' : x \text{ is a } \beta' \text{-approximation for allocation on } \Omega'\}.\]

It is straightforward to check that \( x \) is a \( \bar{\beta} \)-approximation for allocation on \( \Omega' \). As \( x \) is a \( \beta \)-approximation for allocation on \( \Omega' \) and \( \Omega' \subseteq \Omega, \bar{\beta} \geq \beta > 0 \). As \( x \) is not ARNIE on \( \Omega' \), \( x \) is not optimal on \( \Omega' \), so \( \bar{\beta} < 1 \).

Let \( \{\tilde{v}^k_i\}_{k=1}^\infty \) denote a sequence such that \( \tilde{v}^k > 0 \) and \( \lim_{k \to \infty} \tilde{v}^k = 0 \). For all \( k \), there exists \( \tilde{v}^k_i \in [\min\{v_i, \tilde{v}_i\}, \max\{v_i, \tilde{v}_i\}] \) such that \((\bar{\beta} + \epsilon^k)W^*(\tilde{v}^k_i) > W_x(\tilde{v}^k_i)\). The sequence \( \{\tilde{v}^k_i, W_x(\tilde{v}^k_i), W^*(\tilde{v}^k_i)\}_{k=1}^\infty \) is bounded. Thus, by the Bolzano–Weierstrass theorem, we can pick subsequences \( \{\epsilon^k\}_{k=1}^\infty \) and \( \{v^k_i\}_{k=1}^\infty \) such that all three converge, where we denote \( v^\infty_i = \lim_{k \to \infty} v^k_i, \sigma^\infty_x = \lim_{k \to \infty} W_x(v^k_i), \) and \( \sigma^\infty_{OPT} = \lim_{k \to \infty} W^*(v^k_i) \). As for all \( k \),

\[
\bar{\beta}W^*(v^k_i) \leq W_x(v^k_i) \leq (\bar{\beta} + \epsilon^k)W^*(v^k_i),
\]

Electronic copy available at: https://ssrn.com/abstract=3544100
it follows that $\overline{\beta}\sigma_{\text{OPT}}^\infty = \sigma_x^\infty$.

We will check four cases that are jointly exhaustive, and show that in each case $x$ is not a $\overline{\beta}$-approximation for investment on $\Omega'$.

**Case 1:** Suppose the first clause of Definition 2.9 is not satisfied, so there exists $(v, A)$ and $\tilde{v}_i$ such that $i \in x(v, A)$, $\tilde{v}_i > v_i$, and $W_x(\tilde{v}_i, v_{-i}, A) - W_x(v_i, v_{-i}, A) < \tilde{v}_i - v_i$. Either $\sigma_x^\infty - W_x(v_i) < v_i^\infty - v_i$, or $W_x(\tilde{v}_i) - \sigma_x^\infty < \tilde{v}_i - v_i^\infty$.

**Case 1a:** Suppose $\sigma_x^\infty - W_x(v_i) < v_i^\infty - v_i$.

If $v_i^\infty = v_i$, then by lower semi-continuity, we have $\sigma_x^\infty - W_x(v_i) \geq 0$, a contradiction. Thus, $v_i^\infty > v_i$.

Consider the binary investment technology $I^k_i = \{(v_i, 0), (v^k_i, v_i^k - v_i)\}$. Observe that

$$W_x(I^k_i) \leq W_x(v^k_i) - (v^k_i - v_i)$$

$$W^*(I^k_i) \geq W^*(v^k_i) - (v^k_i - v_i).$$

Hence,

$$\overline{\beta} \lim_{k \to \infty} W^*(I^k_i) \geq \overline{\beta} (\sigma_{\text{OPT}}^\infty - (v_i^\infty - v_i)) > \sigma_x^\infty - (v_i^\infty - v_i) \geq \lim_{k \to \infty} W_x(I^k_i).$$

**Case 1b:** Suppose $W_x(\tilde{v}_i) - \sigma_x^\infty < \tilde{v}_i - v_i^\infty$.

Consider the binary investment technology $I^k_i = \{(v^k_i, 0), (\tilde{v}_i, \tilde{v}_i - v_i^k)\}$. Observe that

$$W_x(I^k_i) \leq W_x(\tilde{v}_i) - (\tilde{v}_i - v_i^k)$$

$$W^*(I^k_i) \geq W^*(v_i^k).$$

Hence,

$$\overline{\beta} \lim_{k \to \infty} W^*(I^k_i) \geq \overline{\beta} \sigma_{\text{OPT}}^\infty = \sigma_x^\infty > W_x(\tilde{v}_i) - (\tilde{v}_i - v_i^\infty) \geq \lim_{k \to \infty} W_x(I^k_i).$$

**Case 2:** Suppose Clause 2 of Definition 2.9 is not satisfied, so that

1. $i \notin x(v, A)$;
2. $\tilde{v}_i < v_i$;
3. for all $\epsilon > 0$, we have $W^*(v_i - \epsilon) < W^*(v_i)$; and
4. $W_x(\tilde{v}_i) - W_x(v_i) < 0$.

There are two cases to consider; either $v_i^\infty < v_i$ or $v_i^\infty = v_i$.

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Case 2a: Suppose \( v_i^\infty < v_i \). Consider the technology \( I^k_i = \{(v^k_i, 0), (v_i, 0)\} \).

\[
\mathcal{W}_x(I^k_i) \leq W_x(v_i^k) \\
\mathcal{W}^*(I^k_i) \geq W^*(v_i).
\]

As for all \( \epsilon > 0 \) we have \( W^*(v_i - \epsilon) < W^*(v_i) \), it follows that

\[
W^*(v_i) > W^*(v_i^\infty).
\]

Thus,

\[
\lim_{k \to \infty} \mathcal{W}^*(I^k_i) \geq \beta W^*(v_i) > \beta W^*(v_i^\infty) = W_x(v_i^\infty) \geq \lim_{k \to \infty} \mathcal{W}_x(I^k_i).
\]

Case 2b: Suppose \( v_i^\infty = v_i \). Let \( I^k_i = \{(\tilde{v}_i, 0), (v_i, 0)\} \).

\[
\mathcal{W}_x(I^k_i) \leq W_x(\tilde{v}_i) \\
\mathcal{W}^*(I^k_i) \geq W^*(v_i^k).
\]

By lower semi-continuity, we have

\[
\sigma^\infty_x = \lim_{k \to \infty} W_x(v_i^k) \geq W_x \left( \lim_{k \to \infty} v_i^k \right) = W_x(v_i^\infty) = W_x(v_i).
\]

Thus,

\[
\lim_{k \to \infty} \mathcal{W}^*(I^k_i) \geq \beta \sigma^\infty_{\text{OPT}} = \sigma^\infty_x \geq W_x(v_i) \geq W_x(\tilde{v}_i) = \lim_{k \to \infty} \mathcal{W}_x(I^k_i).
\]

Proof of Theorem 2.4

As before, let \( (v^t_n, c^1_n) \) denote an arbitrary element of \( \arg\max_{(v_n, c_n) \in I_n} \{v_n - c_n\} \), and let \( (v^1_n, c^1_n) \) denote a costless investment \( (c^1_n = 0) \). We suppress the dependence of functions on \( A \).

Consider the allocation \( x(v^t - c^1) \). We now construct an investment profile by requiring all bidders in this allocation to invest \( (v^t_n, c^1_n) \), and all other bidders to invest \( (v^1_n, c^1_n) \). Formally,
let \((\hat{v}, \hat{c})\) be the investment profile such that, for all \(n\),

\[
(\hat{v}_n, \hat{c}_n) = \begin{cases} 
(v^+_n, c^+_n) & \text{if } n \in x(v^+ - c^+) \\
(v^-_n, c^-_n) & \text{otherwise} 
\end{cases}
\]

Recall that the threshold price for bidder \(n\) at instance \((v, A)\) is

\[
t^*_n(v, A) = \{\inf \tilde{v}_n : n \in x(\tilde{v}_n, v_n - A) = 1 \text{ and } (\tilde{v}_n, v_n - A) \in \Omega\}.
\]

Suppressing \(A\), let \(t^*(v)\) be the profile of threshold prices at \((v, A)\).

**Lemma A.2.** Let \(v^k\) be the value profile with the first \(|N| - k\) elements equal to the corresponding elements of \(v^+ - c^+\), and the last \(k\) elements equal to the corresponding elements of \(\hat{v}\). For all \(k \in \{0, 1, \ldots, |N|\}\), \(x(v^k) = x(v^+ - c^+)\).

**Proof.** We argue by induction. By definition, \(x(v^0) = x(v^+ - c^+)\). Suppose \(x(v^k) = x(v^+ - c^+)\). Moving from \(v^k\) to \(v^{k+1}\) either raises the value of a bidder in \(x(v^k)\) or lowers the value of a bidder not in \(x(v^k)\). Thus, by \(x\) monotone and non-bossy, the \(x(v^{k+1}) = x(v^k) = x(v^+ - c^+)\); this proves Lemma A.2. □

**Lemma A.3.** If \(x\) is monotone and non-bossy, then for all \((v, A)\) and \(\tilde{v}_n\), if

1. Either: \(\tilde{v}_n \geq v_n\) and \(x_n(v, A) = 1\)

2. Or: \(\tilde{v}_n \leq v_n\) and \(x_n(v, A) = 0\)

then for all \(m \neq n\) and all \(\tilde{v}_m\) such that \(x_m(\tilde{v}_m, v_m - A) = x_m(v, A)\):

\[
x_m(v, A) = x_m(\tilde{v}_n, \tilde{v}_m, v_n - \{nm\}, A).
\]

**Proof.** By \(x\) non-bossy,

\[
x_n(\tilde{v}_m, v_m - A) = x_n(v, A).
\]

By the previous equation and \(x\) monotone,

\[
x_n(\tilde{v}_n, \tilde{v}_m, v_n - \{nm\}, A) = x_n(\tilde{v}_m, v_m - A).
\]

By the previous equation and \(x\) non-bossy,

\[
x_m(\tilde{v}_n, \tilde{v}_m, v_n - \{nm\}, A) = x_m(\tilde{v}_m, v_m - A).
\]

which proves Lemma A.3. □
Lemma A.4. If $x$ is monotone and non-bossy, then $t_n^x(v^\uparrow - c^\uparrow) \geq t_n^x(\hat{v})$ for $n \in x(v^\uparrow - c^\uparrow)$ and $t_n^x(v^\downarrow - c^\downarrow) \leq t_n^x(\hat{v})$ for $n \notin x(v^\uparrow - c^\downarrow)$.

Proof. We argue by induction. Let value profile $v^k$ be as defined as in Lemma A.2. The inductive hypothesis is: $t_n^x(v^\uparrow - c^\uparrow) \geq t_n^x(v^k)$ for $n \in x(v^\uparrow - c^\uparrow)$ and $t_n^x(v^\downarrow - c^\downarrow) \leq t_n^x(v^k)$ for $n \notin x(v^k)$.

The hypothesis holds by definition for $k = 0$. Suppose it holds for some $k$. By Lemma A.2, $x(v^k) = x(v^\uparrow - c^\uparrow)$. Moving from $v^k$ to $v^{k+1}$ either raises the value of a bidder in $x(v^k)$ or lowers the value of a bidder not in $x(v^k)$. By the inductive hypothesis for $k$ and Lemma A.3, $t_n^x(v^\uparrow - c^\uparrow) \geq t_n^x(v^k) \geq t_n^x(v^{k+1})$ for $n \in x(v^\uparrow - c^\uparrow)$ and $t_n^x(v^\downarrow - c^\downarrow) \leq t_n^x(v^k) \leq t_n^x(v^{k+1})$ for $n \notin x(v^\uparrow - c^\downarrow)$. Thus the inductive hypothesis holds for $k + 1$. This completes the proof of Lemma A.4.

Lemma A.5. $(\hat{v}, \hat{c})$ is a Nash equilibrium of the investment game $(I, A)$ facing threshold auction $(x, p^x)$.

Proof. By Lemma 2.2, it suffices to check that bidders choosing $(v_n^\uparrow, c_n^\downarrow)$ cannot profitably deviate to $(\hat{v}_n^\uparrow, c_n^\downarrow)$ and vice versa. (Recall that $c_n^\downarrow = 0$.)

Suppose that under $(\hat{v}, \hat{c})$, $n$ plays $(v_n^\uparrow, c_n^\downarrow)$, so $n \in x(v^\uparrow - c^\downarrow)$. Then

$$\max \{v_n^\uparrow - t_n^x(\hat{v}), 0\} - c_n^\downarrow \geq \max \{v_n^\uparrow - t_n^x(v^\uparrow - c^\downarrow), 0\} - c_n^\downarrow \geq 0,$$

where the first inequality is by Lemma A.4 and the second inequality is by $n \in x(v^\uparrow - c^\downarrow)$. This implies:

$$\max \{v_n^\uparrow - t_n^x(\hat{v}), 0\} - c_n^\downarrow = \max \{v_n^\uparrow - c_n^\downarrow - t_n^x(\hat{v}), 0\} \geq \max \{v_n^\uparrow - c_n^\downarrow - t_n^x(\hat{v}), 0\} - c_n^\downarrow.$$

The left-hand side is $n$’s utility from playing $(v_n^\uparrow, c_n^\downarrow)$ and the right-hand side is $n$’s utility from playing $(v_n^\uparrow, c_n^\downarrow)$. Hence, $n$ cannot profit by deviating to $(v_n^\uparrow, c_n^\downarrow)$.

Suppose that under $(\hat{v}, \hat{c})$, $n$ plays $(v_n^\uparrow, c_n^\downarrow)$, so $n \notin x(v^\uparrow - c^\downarrow)$. Then we have

$$\max \{v_n^\uparrow - t_n^x(\hat{v}), 0\} - c_n^\downarrow \leq \max \{v_n^\uparrow - t_n^x(v^\uparrow - c^\downarrow), 0\} - c_n^\downarrow \leq 0 \leq \max \{v_n^\uparrow - t_n^x(\hat{v}), 0\} - c_n^\downarrow,$$

where the first inequality is by Lemma A.4 and the second inequality is by $n \notin x(v^\uparrow - c^\downarrow)$. The left-hand side is $n$’s utility from deviating to $(v_n^\uparrow, c_n^\downarrow)$ and the right-hand side is $n$’s utility from playing $(v_n^\uparrow, c_n^\downarrow)$. Hence, $n$ cannot profit by deviating to $(v_n^\uparrow, c_n^\downarrow)$; this proves Lemma A.5.
Lemma A.6. If \( x \) is monotone, non-bossy, and a \( \beta \)-approximation for allocation, then

\[
W_x(\hat{v}, A) - \sum_n \hat{c}_n \geq \beta \max_{(v,c) \in I} \left\{ W^*(v, A) - \sum_n c_n \right\}.
\]  

(14)

Proof. Let \((v^*, c^*)\) be a profile of investments that attains the maximum on the right-hand side of (14). By Lemma A.2, \( x(\hat{v}) = x(v^+ - c^+) \). Recall that, by construction,

\[
(\hat{v}_n, \hat{c}_n) = \begin{cases} 
(v^+_n, c^+_n) & \text{if } n \in x(v^+ - c^+) \\
(v^+_n, c^+_n) & \text{otherwise}
\end{cases}
\]

Hence,

\[
W_x(\hat{v}) - \sum_n \hat{c}_n = w(x(\hat{v}) | \hat{v}) - \sum_n \hat{c}_n = w(x(v^+ - c^+) | \hat{v}) - \sum_n \hat{c}_n = W_x(v^+ - c^+)
\]

\[
\geq \beta W^*(v^+ - c^+) \geq \beta W^*(v^* - c^*) \geq \beta \left( W^*(v^*) - \sum_n c^*_n \right);
\]

this proves Lemma A.6.

Combining Lemmata A.5 and A.6 completes the proof.

Proof of Lemma 3.3

We begin with a general lemma on submodular functions.

Lemma A.7. Let \( q : \wp(G) \to \mathbb{R}_+ \) be a non-negative submodular function, i.e. for all \( F', F'' \subseteq G \):

\[
q(F' \cup F'') + q(F' \cap F'') \leq q(F') + q(F'').
\]

For all \( F \subseteq G \), there exists an additive value function \( \alpha^* : G \to \mathbb{R}_+ \) such that \( \alpha^*(F) = q(F) \) and for all \( F' \), \( \alpha^*(F') \leq q(F') \).

Proof. All submodular functions are fractionally sub-additive (Lehmann et al., 2006). Thus, there exists a family of additive value functions \((\alpha^l)_{l \in L}\) such that for all \( F' \), \( q(F') = \max_l \alpha^l(F') \).

Fix some arbitrary \( F \). Let \( \alpha^* \in \arg\max_{\alpha^l \in L} \{ \alpha^l(F) \} \). \( \alpha^*(F) = q(F) \), and for all \( F' \), \( \alpha^*(F') \leq q(F') \).
Now, we can develop the proof of Lemma 3.3: For any $F \subseteq G$, let

$$v^F_i \equiv \operatorname{argmax}_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\}$$

By $v^F_i \in \text{XOS}$, there exists a family of additive value functions $(\alpha^l)_{l \in L}$ such that $v^F_i = \max_{l \in L} \alpha^l$. Let $\tilde{\alpha}^F = \operatorname{argmax}_{\alpha^l \in L} \{\alpha^l(F)\}$. We now define another additive value function $\alpha^F$ as follows:

$$\alpha^F_g \equiv \begin{cases} \tilde{\alpha}^F_g & \text{if } g \in F \\ 0 & \text{otherwise.} \end{cases}$$

By $c_i$ isotone,

$$\max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\} \leq \tilde{\alpha}^F(F) - c_i(\tilde{\alpha}^F) \leq \alpha^F(F) - c_i(\alpha^F).$$

$\alpha^F \in \text{XOS}$, so

$$\max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\} = \alpha^F(F) - c_i(\alpha^F).$$

The next step is to define, for each set of goods $F$, an additive value function $\overline{\alpha}^F$ that divides the cost $c_i(\alpha^F)$ appropriately across the various goods in $F$.

For any $F, F'$, let $\alpha^{F \triangledown F'}$ be the additive value function defined by:

$$\alpha^{F \triangledown F'}_g \equiv \begin{cases} \alpha^F_g & \text{if } g \in F' \\ 0 & \text{otherwise.} \end{cases}$$

Fix some arbitrary $F$. Let $q^F : \wp(G) \to \mathbb{R}$ be the function defined by

$$q^F(F') \equiv \alpha^{F \triangledown F'}(F') - c_i(\alpha^{F \triangledown F'})$$

(for all $F'$). As $c_i$ is supermodular on additive valuations, the function $q^F(\cdot)$ is submodular. Moreover, by submodularity of $q^F$, it follows that for all $F'$ we have:

$$q^F(F') + q^F(G \setminus F') \geq q^F(F' \cup (G \setminus F')) + q^F(F' \cap (G \setminus F')).$$

(15)
Moreover, we have

\[ q^F(G \setminus F') = \alpha^{F_b(G \setminus F')}(G \setminus F') - c_i(\alpha^{F_b(G \setminus F')}) \]

\[ = \alpha^{F_b(G \setminus F')}(F) - c_i(\alpha^{F_b(G \setminus F')}) \]

\[ \leq \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\} \]

\[ = \alpha^F(F) - c_i(\alpha^F). \]

Rearranging terms in (15) yields

\[ q^F(F') \geq \alpha^F(F) - c_i(\alpha^F) - q^F(G \setminus F') \geq 0. \]

Thus, \( q^F \) is a non-negative submodular function. By Lemma A.7, we can find an additive value function \( \overline{\alpha}^F \) such that \( \overline{\alpha}^F(F) = q^F(F) \) and for all \( F' \), \( \overline{\alpha}^F(F') \leq q^F(F') \).

We assert now that the maximum of the family of additive value functions so constructed is exactly equal to the pivotal value function \( \overline{v}_i \), that is, for all \( F \),

\[ \max_{F' \in \mathcal{P}(G)} \left\{ \overline{\alpha}^F(F) \right\} = \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\} \equiv \overline{v}_i(F). \]

By construction, for all \( F \),

\[ \overline{\alpha}^F(F) = q^F(F) = \alpha^F(F) - c_i(\alpha^F) = \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\}. \]

which implies that for all \( F \),

\[ \max_{F' \in \mathcal{P}(G)} \left\{ \overline{\alpha}^F(F) \right\} \geq \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\}. \]

Also by construction, for all \( F \) and \( F' \),

\[ \overline{\alpha}^{F'}(F) \leq q^{F'}(F) = \alpha^{F_b(F')} - c_i(\alpha^{F_b(F')}) \leq \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\}, \]

which implies that for all \( F \),

\[ \max_{F' \in \mathcal{P}(G)} \left\{ \overline{\alpha}^{F'}(F) \right\} \leq \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\}. \]

Thus, for all \( F \),

\[ \max_{F' \in \mathcal{P}(G)} \left\{ \overline{\alpha}^{F'}(F) \right\} = \max_{v_i \in \text{XOS}} \{v_i(F) - c_i(v_i)\} \equiv \overline{v}_i(F); \]
we conclude that $\pi_i \in XOS$. 